Homework 2

Due: Wednesday, February 4

Throughout this problem set, let V be a vector space over a field $\mathbb{F}$.

1. Suppose $A \subseteq V$ and that $0 \in A$. Show that A is a linearly dependent set.

2. Let $v_1, \cdots, v_m \in V$ be a linearly independent list of vectors. Suppose that $v \in V$. Prove (directly from the definition, without using results from class) that $\{v_1, \cdots, v_m, v\}$ is linearly independent if and only if $v \notin \operatorname{span}(v_1, \cdots, v_m)$.

3. Let $\{v_1, \cdots, v_m\}$ be a basis for V. Suppose $v \in V$.

   (a) Show that there is *at least one* choice of numbers $a_1, \cdots, a_m \in \mathbb{F}$ such that

   $$v = \sum_{i=1}^{m} a_i v_i.$$

   (HINT: *A basis spans.*)

   (b) Show that there is *exactly one* choice of numbers $a_1, \cdots, a_m \in \mathbb{F}$ such that $v = \sum_{i=1}^{m} a_i v_i$. (HINT: *A basis is linearly independent.*)

4. [KK] 1.3.5.

5. Consider $\mathbb{R}[x]$ as a vector space over $\mathbb{R}$.

   (a) Let

   $$A = \{1, x, x^2, x^3, \cdots\} = \{x^n : n \in \{0, 1, 2, 3 \cdots\}\} \subset \mathbb{R}[x].$$

   Show that A is a basis for $\mathbb{R}[x]$.

   (b) Let $B = \{f_1, \cdots, f_m\} \subset \mathbb{R}[x]$ be any *finite* subset. Show that B is not a basis for $\mathbb{R}[x]$. (HINT: *Degree!*)

Extra: Is $\mathbb{R}$ finite-dimensional over $\mathbb{Q}$? Is $\mathbb{C}$ finite-dimensional over $\mathbb{R}$? Is $\mathbb{C}$ finite-dimensional over $\mathbb{Q}$? ([KK] 1.2.1, 1.3.4)