Homework 2
Due: Wednesday, February 4

Throughout this problem set, let $V$ be a vector space over a field $\mathbb{F}$.

1. Suppose $A \subseteq V$ and that $0 \in A$. Show that $A$ is a linearly dependent set.

2. Let $v_1, \ldots, v_m \in V$ be a linearly independent list of vectors. Suppose that $v \in V$. Prove (directly from the definition, without using results from class) that \( \{v_1, \ldots, v_m, v\} \) is linearly independent if and only if $v \not\in \text{span}(v_1, \ldots, v_m)$.

3. Let \( \{v_1, \ldots, v_m\} \) be a basis for $V$. Suppose $v \in V$.
   
   (a) Show that there is at least one choice of numbers $a_1, \ldots, a_m \in \mathbb{F}$ such that
   
   $$v = \sum_{i=1}^{m} a_i v_i.$$  
   
   (HINT: A basis spans.)

   (b) Show that there is exactly one choice of numbers $a_1, \ldots, a_m \in \mathbb{F}$ such that $v = \sum_{i=1}^{m} a_i v_i$.
   
   (HINT: A basis is linearly independent.)

4. [KK] 1.3.5.

5. Consider $\mathbb{R}[x]$ as a vector space over $\mathbb{R}$.
   
   (a) Let $A = \{1, x, x^2, x^3, \ldots\} = \{x^n : n \in \{0, 1, 2, 3, \ldots\}\} \subseteq \mathbb{R}[x]$.
   
   Show that $A$ is a basis for $\mathbb{R}[x]$.

   (b) Let $B = \{f_1, \ldots, f_m\} \subseteq \mathbb{R}[x]$ be any finite subset. Show that $B$ is not a basis for $\mathbb{R}[x]$.
   
   (HINT: Degree!)

Extra: Is $\mathbb{R}$ finite-dimensional over $\mathbb{Q}$? Is $\mathbb{C}$ finite-dimensional over $\mathbb{R}$? Is $\mathbb{C}$ finite-dimensional over $\mathbb{Q}$? ([KK] 1.2.1, 1.3.4)