Homework 10  
Due: Wednesday, April 22

1. Suppose $S, T \in \mathcal{L}(V)$ and $ST$ is nilpotent. Show that $TS$ is nilpotent. (HINT: Suppose $(ST)^2 = 0$. Can you show that $(TS)^3 = 0$?)

2. (a) Given an example of a linear transformation $S \in \mathcal{L}(\mathbb{R}^3)$ with minimal polynomial $\mu_S(x) = x^2$. What is $\chi_S(x)$?

(b) Extra: Let $V/\mathbb{F}$ be an $n$-dimensional vector space, and suppose $T \in \mathcal{L}(V)$ has minimal polynomial $\mu_T(x) = (x - \lambda)^e$ for some $\lambda \in \mathbb{F}$ and $e \in \mathbb{N}$. Show that $\chi_T(x) = (x - \lambda)^n$.

3. (a) Give an example of a linear transformation $S \in \mathcal{L}(\mathbb{R}^4)$ with minimal and characteristic polynomials

$$\mu_S(x) = x(x - 1)^2(x - 2)$$

$$\chi_S(x) = x(x - 1)^2(x - 2).$$

(b) Give an example of a linear transformation $T \in \mathcal{L}(\mathbb{R}^4)$ with minimal and characteristic polynomials

$$\mu_T(x) = x(x - 1)(x - 2)$$

$$\chi_T(x) = x(x - 1)^2(x - 2).$$

4. Consider the linear transformation $T \in \mathcal{L}(\mathbb{Q}^5)$ whose matrix, in the standard basis, is:

$$[T] = \begin{pmatrix}
5 & 4 & 0 & 0 & 4 & 3 \\
2 & 3 & 1 & 0 & 5 & 1 \\
0 & -1 & 2 & 0 & 2 & 0 \\
-8 & -8 & -1 & 2 & -12 & -7 \\
0 & 0 & 0 & -1 & 0 & \ \\
-8 & -8 & -1 & 0 & -9 & -5
\end{pmatrix}.$$ 

It turns out that

$$\chi_T(x) = x^6 - 6x^5 + 9x^4 + 8x^3 - 24x^2 + 16 = (x + 1)^2(x - 2)^4$$

The dimensions of the generalized eigenspaces are as follows:

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<tr>
<th>$j$</th>
<th>dim ker$((T + 1)^j)$</th>
<th>dim ker$((T - 2)^j)$</th>
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Find the Jordan normal form of $T$. 

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