Homework 10
Due: Wednesday, April 22

1. Suppose $S, T \in \mathcal{L}(V)$ and ST is nilpotent. Show that TS is nilpotent. (HINT: *Suppose* $(ST)^2 = 0$. *Can you show that* $(TS)^3 = 0$*?*)

2. (a) Given an example of a linear transformation $S \in \mathcal{L}(\mathbb{R}^3)$ with minimal polynomial $\mu_S(x) = x^2$. What is $\chi_S(x)$?

   (b) *Extra:* Let $V/\mathbb{F}$ be an n-dimensional vector space, and suppose $T \in \mathcal{L}(V)$ has minimal polynomial $\mu_T(x) = (x-\lambda)^e$ for some $\lambda \in \mathbb{F}$ and $e \in \mathbb{N}$. Show that $\chi_T(x) = (x-\lambda)^n$.

3. (a) Give an example of a linear transformation $S \in \mathcal{L}(\mathbb{R}^4)$ with minimal and characteristic polynomials

$$\mu_S(x) = x(x-1)^2(x-2)$$
$$\chi_S(x) = x(x-1)^2(x-2).$$

   (b) Give an example of a linear transformation $T \in \mathcal{L}(\mathbb{R}^4)$ with minimal and characteristic polynomials

$$\mu_T(x) = x(x-1)(x-2)$$
$$\chi_T(x) = x(x-1)^2(x-2).$$

4. Consider the linear transformation $T \in \mathcal{L}(\mathbb{Q}^5)$ whose matrix, in the standard basis, is:

$$[T] = \begin{pmatrix} 5 & 4 & 0 & 0 & 4 & 3 \\ 2 & 3 & 1 & 0 & 5 & 1 \\ 0 & -1 & 2 & 0 & 2 & 0 \\ -8 & -8 & -1 & 2 & -12 & -7 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ -8 & -8 & -1 & 0 & -9 & -5 \end{pmatrix}.$$

   It turns out that

$$\begin{aligned} \chi_T(x) &= x^6 - 6x^5 + 9x^4 + 8x^3 - 24x^2 + 16 \\ &= (x+1)^2(x-2)^4 \end{aligned}$$

   The dimensions of the generalized eigenspaces are as follows:

j	$\dim\ker((T+1)^j)$	$\dim\ker((T-2)^j)$
1	2	2
2	2	3
3	2	4
4	2	4

   Find the Jordan normal form of T.