Homework 11

Due: Friday, November 9

1. Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree $n \geq 2$. In class, we will show that there exists some number R_0 such that if $|z| > R_0$, then
$$|P(z)| > |z^n| \frac{|a_n|}{2}.$$
For any $R > 0$, let C_R be the simple, positive closed contour around the circle of radius R centered at the origin, and let
$$I_R = \int_{C_R} \frac{1}{P(z)} dz.$$
Suppose that $P(z)$ has all its roots inside the circle $|z| = r$.
   (a) Show that for any $R \geq r$, $I_R = I_r$.
   (b) Show that
   $$\lim_{R \to \infty} I_R = 0.$$
   (c) Show that
   $$\int_{C_r} \frac{1}{P(z)} dz = 0.$$

2. *Continuation of* #1 Now, suppose $P(z) = a_1 z + a_0$ is a polynomial of degree one, and that its root is inside the circle $|z| = r$.
   (a) What is
   $$\int_{C_r} \frac{1}{P(z)} dz?$$
   (b) Why does the argument of Problem 1 not apply in this situation?

3. Let C be the unit circle taken once in the positive direction.
   (a) Explain why the Cauchy integral formula does not directly apply to
   $$\int_C \frac{\mathrm{Re}(z)}{z - 1/2} dz. \tag{1}$$
   (HINT: *You may assume the result of [BC]24.7.*)
   (b) Show that if $|z| = 1$, then $\mathrm{Re}(z) = (z + z^{-1})/2$.
   (c) Use this to evaluate (1).

4. [BC]50.4.

5. [BC]50.2.