Midterm

Friday, October 6

Each problem is worth twelve points, evenly divided among the subproblems. You must justify your answer to receive full credit. You may express complex numbers in rectangular or polar coordinates.

1. For each statement, indicate if it is **TRUE** or **FALSE**, and provide a short (1-2 line) justification.
   (a) If $z_1, z_2 \in \mathbb{C}$, then $\operatorname{Im}(z_1 + z_2) = \operatorname{Im}(z_1) + \operatorname{Im}(z_2)$.
   (b) If $z_1, z_2 \in \mathbb{C}$, then $\operatorname{Re}(z_1 z_2) = \operatorname{Re}(z_1)\operatorname{Re}(z_2)$.
   (c) If $z_1, z_2 \in \mathbb{C}$, then $\operatorname{Arg}(z_1 z_2) = \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2)$.
   (d) $|z| = 1$ if and only if $z^{-1} = \bar{z}$.
2. (a) Compute $(-4\sqrt{2} + i\,4\sqrt{2})^{1/3}$. Graph these numbers.
   (b) Suppose $z_1, \ldots, z_n$ are n complex numbers evenly spaced on a circle of radius r, centered at the origin. Show that $z_1^n = z_2^n = \cdots = z_n^n$.
3. Is the function
   $$g(z) = |z|^2$$
   differentiable? Prove your answer.
4. Prove, *without using trigonometry*, that for any real number θ,
   $$\sin(2\theta) = 2\sin(\theta)\cos(\theta).$$
5. Consider the function $f(z) = z^2$.
   (a) Calculate $f'(z)$, using the definition of the derivative as a limit.
   (b) In this problem, express a number z as $z = x + iy$.
      i. Fix a real number c. Let $S_c = \{z \in \mathbb{C} : \operatorname{Re}(f(z)) = c\}$. Calculate S_c (in terms of x and y).
      ii. Fix a real number d. Let $T_d = \{z \in \mathbb{C} : \operatorname{Im}(f(z)) = d\}$. Calculate T_d.
      iii. Pick a positive value of c and of d. Graph S_c and T_d.

Extra credit *In the situation of* 2(b)*, suppose $P(z)$ is a polynomial of degree $r < n$. What is $\sum_{j=1}^{n} P(z_j)$? Justify.*