Name:________________________ **Final** Tuesday, December 12

Each problem is worth twelve points, evenly divided among the subproblems. You must justify your answer to receive full credit.

1	2	3	4	5	6	7	total

1. *For each question, give a* **brief** *(one or two line) justification.*
   (a) Describe all numbers z such that:
      i. $\overline{z} = iz$.
      ii. $\exp(z) = i$.
   (b) For each of the following functions, state whether that function is or is not analytic.
      i. $f(z) = z$.
      ii. $g(z) = \overline{z}$.
      iii. $h(z) = |z|$.
2. Consider the following two contours

$$\begin{array}{lll} C_1: & z_1(t) = \exp(it) & t \in [0, \pi/2] \\ C_2: & z_2(t) = \exp(-it) & t \in [0, 3\pi/2] \end{array}$$

   and evaluate the following contour integrals.
   (a) $\int_{C_1} z dz$.
   (b) $\int_{C_2} z dz$.
   (c) $\int_{C_1} \overline{z} dz$.
   (d) $\int_{C_2} \overline{z} dz$.
3. Consider the function

$$f(z) = \frac{1}{z^2 - z^4}.$$

   (a) Find a series expansion for f valid on $0 < |z| < 1$.
   (b) Find a series expansion for f valid on $1 < |z| < \infty$.
   (c) Let $g(z) = \frac{\exp(z)-1-z}{z^2-z^4}$. There exists an analytic function $\widetilde{g}(z)$ which is analytic on some neighborhood $N_R(0)$ such that, for each $z \in N_R(0) - \{0\}$, $\widetilde{g}(z) = g(z)$. What is $\widetilde{g}(0)$? Explain.

4. Compute

$$\int_0^\infty \frac{1}{(x^2+1)(x^2+4)}dx.$$

5. Let C be a simple, closed, contour. Suppose that f and g are both analytic on and inside C and that $f = g$ on C. (In other words, for every $z \in C$, $f(z) = g(z)$.) Prove that $f = g$ throughout the interior of C.

6. Suppose that $f(z) = \sum_{n\geq 0} a_n z^n$ converges on some neighborhood of 0.

   (a) Find a series representation for $f(\overline{z})$, in terms of powers of $\overline{z}$.

   (b) Find a series representation for $\overline{f(\overline{z})}$.

   (c) Let $g(z) = \overline{f(\overline{z})}$. Is $g(z)$ analytic? Explain.

7. (a) Suppose f has a zero of order m at a point z_0. Show that

$$\mathrm{res}(\frac{f'(z)}{f(z)}; z_0) = m.$$

   (b) Suppose f is analytic on and inside a simple, positive, closed contour C. Suppose there are points $z_1, \cdots z_r$ inside C such that f has a zero of order m_j at z_j for $j = 1, \cdots, r$, and has no other zeros inside C. What is

$$\int_C \frac{f'(z)}{f(z)}dz?$$

   Justify.

Extra credit Suppose f is analytic on a domain D, and that for each $z \in D$, $z+1$ and $z+i$ are in D, and that $f(z+1) = f(z+i) = f(z)$. Let S be the unit square, with vertices $0, 1, 1+i$, and i.

Suppose that f has m poles and n zeros in the interior of S, and that each pole or zero is simple. Show that $m = n$. (HINT: *Let C be be the simple, positive closed contour around S. Show that* $\int_C \frac{f'(z)}{f(z)}dz = 0$.)