Name:______________________ **Final** Wednesday, December 14

Each problem is worth twelve points, evenly divided among the subproblems. You must justify your answer to receive full credit. You may express complex numbers in rectangular or polar coordinates.

1	2	3	4	5	6	7	8	total

1. Let C be the unit circle $|z| = 1$, parametrized by

$$[0, 2\pi] \xrightarrow{z} \mathbb{C}$$

$$t \longmapsto e^{it}$$

Use the definition of a contour integral to evaluate the following:

(a) $\int_C z dz$.

(b) $\int_C \overline{z} dz$.

(c) $\int_C |z| dz$.

2. (a) For each of the following functions, state whether that function is or is not analytic, and give a brief (one or two line) reason.

i. $f(z) = z$.

ii. $g(z) = \overline{z}$.

iii. $h(z) = |z|$.

(b) Pick one function from part (a) and prove, using the definition of the derivative as a limit, that the function is or is not analytic.

3. Let $\alpha = 1 + i$.

(a) Compute $\mathrm{Log}(\alpha)$ and $\log(\alpha)$.

(b) Find all values of $\alpha^{1/4}$.

(c) Find all values of α^i.

4. Consider the function

$$f(z) = \frac{z}{1 - z^2}.$$

(a) Find a series expansion of $f(z)$ centered at $z = 0$.

(b) Find a series expansion of $f(z)$ centered at $z = 1$.

5. Compute
$$\int_0^\infty \frac{x^2}{(x^2+1)(x^2+9)}dx.$$

6. Let C be a simple, closed contour. Suppose that $f(z)$ and $g(z)$ are analytic on and inside C; that $g(z)$ is never zero on or inside C; and that $|f(z)| \leq |g(z)|$ on C. Prove that $|f(z)| \leq |g(z)|$ in the interior of C.

7. (a) Suppose that $f(z)$ and $g(z)$ each have an isolated singularity at $z = \alpha$, and that $\lambda \in \mathbb{C}$.
   i. Show that $\mathrm{res}(f(z) + g(z); \alpha) = \mathrm{res}(f(z); \alpha) + \mathrm{res}(g(z); \alpha)$.
   ii. Show that $\mathrm{res}(\lambda f(z); \alpha) = \lambda\, \mathrm{res}(f(z); \alpha)$.

   (b) Suppose that $P(z)$ is a polynomial of degree m and that $Q(z)$ is a polynomial of degree n. Let $f(z) = P(z)/Q(z)$, and let C be any simple, closed contour which contains all zeros of P and Q. Prove that
$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)} = m - n.$$
   (HINT: *You may use the fact: If $R(z)$ is the polynomial $\lambda(z-\alpha_1)\cdots(z-\alpha_r)$, then $R'(z)/R(z) = 1/(z-\alpha_1) + \cdots + 1/(z-\alpha_r)$.*)

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8. Let $f(z)$ be a function which is analytic everywhere.

   (a) Let C be a simple, positive, closed contour. Suppose that a and b are distinct points in the interior of C. What is
$$\int_C \frac{f(z)}{(z-a)(z-b)}dz?$$

   (b) Let C_R be the circle of radius R centered at the origin. Suppose that $R > 2\max(|a|, |b|)$, and that $|f(z)| < M$ for all $z \in C_R$. Show that
$$\left|\frac{f(z)}{(z-a)(z-b)}\right| < \frac{4M}{R^2}.$$

   (c) Further suppose $|f(z)| < M$ everywhere. Prove Liouville's theorem: For every $a, b \in \mathbb{C}$, $f(a) = f(b)$.