Fix complex numbers α and β. The set of all z satisfying

$$|z-\alpha| = |z-\beta|$$

is a line in the complex plane. More precisely, it's the perpendicular bisector of the line segment $L_{\alpha\beta}$.

What if we consider

$$\frac{|z-\alpha|}{|z-\beta|} = \lambda \tag{1}$$

for some other positive value of λ?

LEMMA The equation (1) defines the circle of radius

$$r(\alpha,\beta,\lambda) = \left|\frac{\lambda}{1-\lambda^2}\right| |\alpha-\beta| \tag{2}$$

and center

$$c(\alpha,\beta,\lambda) = \frac{\alpha-\lambda^2\beta}{1-\lambda^2} \tag{3}$$

PROOF In general, since $|\gamma|^2 = \gamma\overline{\gamma}$, we have

$$\begin{aligned}
|\gamma-\delta|^2 &= (\gamma-\delta)\overline{(\gamma-\delta)} \\
&= (\gamma-\delta)(\overline{\gamma}-\overline{\delta}) \\
&= \gamma\overline{\gamma} - \delta\overline{\gamma} - \gamma\overline{\delta} + \delta\overline{\delta} \\
&= |\gamma|^2 - 2\operatorname{Re}(\gamma\overline{\delta}) + |\delta|^2
\end{aligned}$$

Therefore, given equation (1), we have

$$\begin{aligned}
|z-\alpha|^2 &= \lambda^2|z-\beta|^2 \\
|z|^2 - 2\operatorname{Re}(\alpha\overline{z}) + |\alpha|^2 &= \lambda^2(|z|^2 - 2\operatorname{Re}(\beta\overline{z}) + |\beta|^2)
\end{aligned}$$

Since $\operatorname{Re}(\gamma+\delta) = \operatorname{Re}(\gamma) + \operatorname{Re}(\delta)$,

$$\begin{aligned}
(1-\lambda^2)|z|^2 - 2\operatorname{Re}((\alpha-\lambda^2\beta)\overline{z}) &= -|\alpha|^2 + \lambda^2|\beta|^2 \\
|z|^2 - 2\operatorname{Re}(\frac{\alpha-\lambda^2}{1-\lambda^2}\overline{z}) + \left|\frac{\alpha-\lambda^2\beta}{1-\lambda^2}\right|^2 &= \left|\frac{\alpha-\lambda^2\beta}{1-\lambda^2}\right|^2 - \frac{|\alpha|^2-\lambda^2|\beta|^2}{1-\lambda^2}
\end{aligned}$$

We then have

$$\left|z - \frac{\alpha - \lambda^2\beta}{1-\lambda^2}\right|^2 = \frac{1}{|1-\lambda^2|^2}(\left|\alpha - \lambda^2\beta\right|^2 - (1-\lambda^2)(|\alpha|^2 - \lambda^2|\beta|^2))$$
$$= \frac{1}{|1-\lambda^2|^2}(|\alpha|^2 - 2\operatorname{Re}(\overline{\alpha}\lambda^2\beta) + \lambda^4|\beta|^2 - (|\alpha|^2 - \lambda^2|\alpha|^2 - \lambda^2|\beta|^2 + \lambda^4|\beta|^2))$$

But since $\lambda \in \mathbb{R}$, $\operatorname{Re}(\lambda\gamma) = \gamma\operatorname{Re}(\lambda)$, so

$$= \frac{1}{1-\lambda^{2^2}}(\lambda^2|\alpha|^2 + \lambda^2|\beta|^2 - 2\lambda^2\operatorname{Re}(\overline{\alpha}\beta))$$
$$\left|z - \frac{\alpha - \lambda^2\beta}{1-\lambda^2}\right|^2 = \frac{\lambda^2}{|1-\lambda^2|^2}|\alpha - \beta|$$

Therefore, (1) defines the circle of radius $r(\alpha, \beta, \lambda)$ and center $c(\alpha, \beta, \lambda)$, as advertised. □

REMARK The shapes defined by

$$\frac{|z-\alpha|}{|z-\beta|} = \lambda$$

and

$$\frac{|z-\beta|}{|z-\alpha|} = \frac{1}{\lambda}$$

are clearly the same. Fortunately, from equations (3) and (2) we have

$$c(\alpha, \beta, \lambda) = c(\beta, \alpha, 1/\lambda)$$
$$r(\alpha, \beta, \lambda) = r(\beta, \alpha, 1/\lambda)$$

so that, independently of which initial form of the equation we choose, we obtain the same circle.

REMARK Consider the special case where $\alpha = 1$ and $\beta = 0$. Suppose $0 < \lambda < 1$. Then we have

$$c(1, 0, \lambda) = \frac{1}{1-\lambda^2}$$
$$r(1, 0, \lambda) = \frac{\lambda}{1-\lambda^2}$$

Note that as λ gets closer to one, both the center and the origin of the circle get larger and larger.

Moreover, the circle intersects the x-axis at two points; one of them is

$$\frac{1}{1-\lambda^2} - \frac{\lambda}{1-\lambda^2} = \frac{1}{1+\lambda};$$

as λ gets ever-closer to 1, this point of intersection gets closer to the midpoint of the line segment between 0 and 1.