Homework 3

Due: Friday, September 8

1. [BC]9.1, 9.2.

2. [BC]9.5.

3. Suppose that ζ is an n^{th} root of unity, but $\zeta \neq 1$.

   (a) Prove that
   $$1 + \zeta + \zeta^2 + \zeta^3 + \cdots + \zeta^{n-1} = 0.$$

   (b) Let m be a natural number, and suppose that $\gcd(m, n) = 1$. Show that
   $$\zeta^m \neq 1.$$

   (c) Let m be a natural number, and suppose that $\gcd(m, n) = 1$. Show that
   $$1 + \zeta^m + \zeta^{2m} + \zeta^{3m} + \cdots + \zeta^{(n-1)m} = 0.$$

4. Give two different proofs of the following statement:

   If $|z| = 1$, then $\overline{z} = z^{-1}$.

5. This problem concerns the equation
   $$(z+1)^{10} = (z-1)^{10}. \tag{1}$$

   (a) Show directly that if z is a solution to Equation (1), then z must also satisfy
   $$|z+1| = |z-1|. \tag{2}$$
   What geometric object does Equation (2) describe?
   Note: We are not *saying that any solution to* (2) *also satisfies* (1)*!*

   (b) Divide both sides of Equation (1) by $(z-1)^{10}$, so that the equation is of the form $w^{10} = 1$. Use this to find all solutions z to Equation 1.

   (c) For each solution z to Equation (1), compute $\overline{z}$.
   (HINT: *Use the result of problem 4.*)