Chapter Nine: Music, Chords and Harmony

If the stars and planets are the gears of the universe, revolving in intricate ways in the skies, then music came to be seen from ancient times as a subtle reflection of this machinery, connecting it to the emotions and to the soul. The link was through the strange integral relationships, which they exhibit. In the case of the sky, we have wheels turning, the cycle of the day, of the month (from one full moon to the next), the year (the time from one vernal equinox to the next, i.e. from one season to the one next year). Integers appear when the cycles are compared, thus there seem to be 29 days in the lunar month (time from one full moon to the next), 365 days in a year (time from one vernal equinox to another, i.e. from one season to the next). But when more careful observations are made, the relations are more complex: there are really 29 ½ days in the lunar month or better 29 days 12 ¾ hours or better …. Likewise the year, not really 365 days but 365 ¼ days in a year, or better 365 days, 6 hours *less* about 11 minutes or ….Gears indeed.

What many early peoples noted was that when strings were plucked producing music, the sounds produced pleasing chords and tunes if the length of the strings had a proportion given by small integers, 2:1, 3:2, 4:3, 5:3, etc. Thus the quality of a tune made by plucking one or more strings was crucially affected by the ratio of the lengths of the string at the times plucked. The same went for blowing into or across holes in pipes and the pipe lengths. These relationships were apparently of great importance to Pythagoras (ca. 560-480 BCE), to the religious cult he started and to his later followers (though nothing really reliable is known about Pythagoras). The Pythagorean School divided up the areas of study into the *quadrivium*, the 4 subjects

- arithmetic
- geometry
- music
- astronomy

all of which contained number, the essence of the regularities of nature, all of which displayed the beauty of the universe. Put simply, even from our modern jaded perspective, is it not startling that strings with simple arithmetic ratios are exactly those which produce beautiful chords? Fortunately or unfortunately, there is a pretty simple explanation, which this Chapter will explain.

Moving ahead in history, this connection of integers with music was of great interest to Galileo also. He starts with

> Salviati: *Impelled by your queries I may give you some of my ideas concerning certain problems in music, a splendid subject, upon which so many eminent men have written: among these is Aristotle himself who has discussed numerous interesting acoustical questions. Accordingly, if on the basis of some easy and tangible experiments, I shall explain some striking phenomena in the domain of sound, I trust my explanations shall meet your approval.*
> Sagredo: *I shall receive them not only gratefully but eagerly. For, although I take pleasure in every kind of musical instrument and have paid considerable attention to harmony, I have never been able to fully understand why some combinations of*

musical tones are more pleasing than others, or why certain combinations not only fail to please but are even highly offensive.

Galileo knew, of course, that all music was produced by rapid vibrations of strings, or air in pipes and sought to make analogies with other oscillating systems, especially his favorite, the pendulum.

If one bows the base string on a viola rather smartly and brings near it a goblet of fine, thin glass having the same tone [*tuono*] as that of the string, this goblet will vibrate and audibly resound. That the undulations of the medium are widely dispersed about the sounding body is evinced by the fact that a glass of water may be made to emit a tone merely by the friction of the finger-tip upon the rim of the glass; for in this water is produced a series of regular waves. The same phenomenon is observed to better advantage by fixing the base of the goblet upon the bottom of a rather large vessel of water filled nearly to the edge of the goblet; for if, as before, we sound the glass by friction of the finger, we shall see ripples spreading with the utmost regularity and with high speed to large distances about the glass. I have often remarked, in thus sounding a rather

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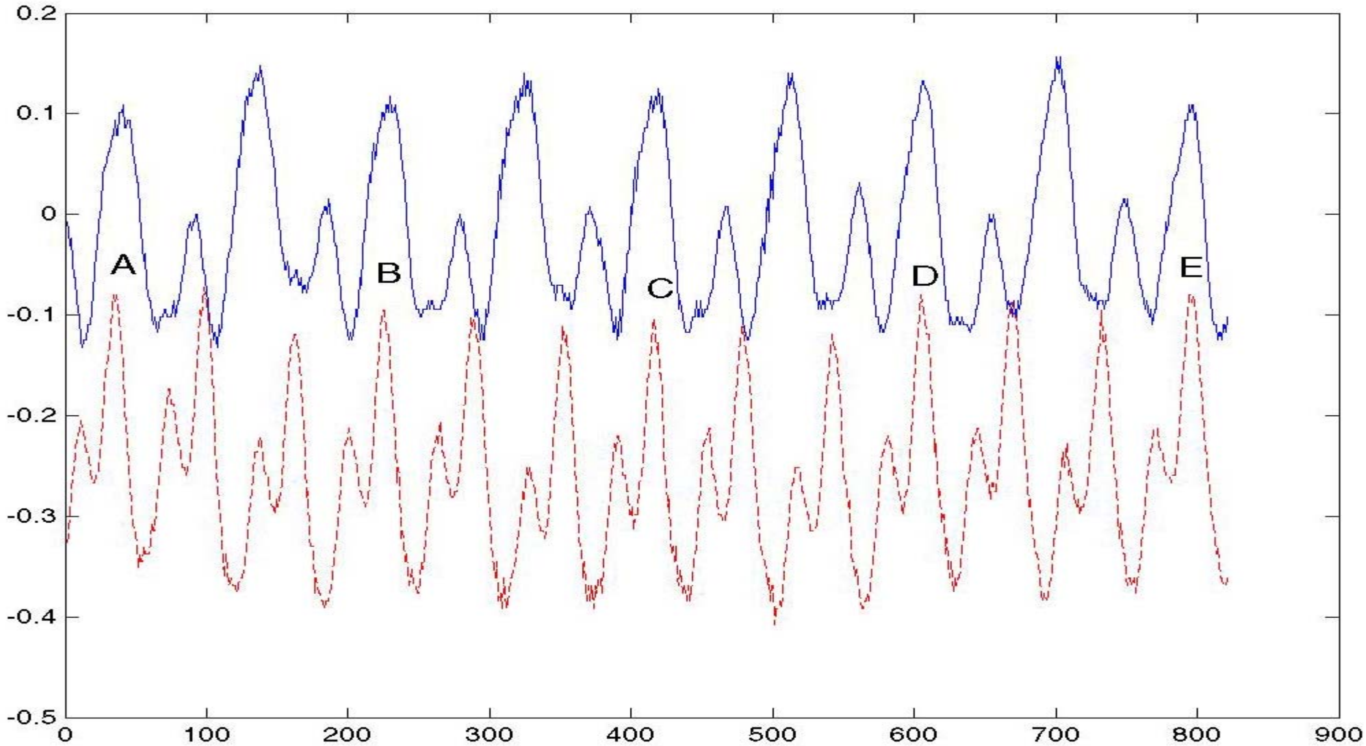
large glass nearly full of water, that at first the waves are spaced with great uniformity, and when, as sometimes happens, the tone of the glass jumps an octave higher I have noted that at this moment each of the aforesaid waves divides into two; a phenomenon which shows clearly that the ratio involved in the octave [*forma dell' ottava*] is two.

SAGR. More than once have I observed this same thing, much to my delight and also to my profit. For a long time I have been perplexed about these different harmonies since the explanations hitherto given by those learned in music impress me as not sufficiently conclusive. They tell us that the diapason, i. e. the octave, involves the ratio of two, that the diapente which we call the fifth involves a ratio of 3:2, etc.; because if the open string of a monochord be sounded and afterwards a bridge be placed in the middle and the half length be sounded one hears the octave; and if the bridge be placed at 1/3 the length of the string, then on plucking first the open string and afterwards 2/3 of its length the fifth is given; for this reason they say that the octave depends upon the ratio of two to one [*contenuta tra'l due e l'uno*] and the fifth upon the ratio of three to two. This explanation does not impress me as sufficient to

On the right, Salviati is discussing his ideas about music and how, since it consists in vibrations, musical sounds from one object can excite another object into vibration. He discusses a specific set up in which a glass is placed in a large vessel, which is then filled nearly to the brim of the glass: the purpose is be able to see the vibration as waves in the water. Then he gets to the key point: if the tone changes from one note to another an octave higher, suddenly you see twice as many water waves, i.e. the frequency has doubled. Then he goes on to what musicians call the *fifth*, the note produced by a string $2/3^{rd}$'s the length of the original. But Sagredo is not convinced!

Well, why not jump ahead in time and look at what the air actually does when music is heard? Edison learned how to pick up the vibrations of air on a flexible membrane and, by fixing a small piece of iron to the membrane, transform the air pressure vibrations into vibrating electrical signals. Then, of course, we can put them in a computer and analyze them anyway we want. I recorded the voice of a female singer singing the major scale, *do, re, mi, fa, sol, la, ti, do* and here is a small part of this recording, showing *do* and *sol*:



I have drawn the vibration of *do* as a solid blue line oscillating around 0; and I moved *sol* down making it a dashed red line oscillating around -.25 simply in order to separate the two curves. Several things are immediately apparent: first of all, these waves are *not* sinusoidal! They are complex and fairly close to being periodic but not exactly periodic either. However, the blue curve for *do* shows 9 periods with major peaks interspersed with minor peaks, while the red curve shows 13 peaks. Look at the points marked A,B,C,D and E. At each letter both curves have peaks but between each pair, there is one extra peak for *do* and two for *sol*. In other words, two periods of *do* match three periods of *sol*. This is the 3:2 correspondence, which was discovered empirically by prehistoric musicians.

What we see is that the vibrations of the chord *do-sol* merge together into one shape that repeats itself every two periods of *do* and every three periods of *sol*. This is exactly what Galileo also claimed, as he describes on the next page, taken a few pages after the previous quote. Note that he guesses that the music consists in *pulses* of airwaves. I think he would have been thrilled to see the actual signals in the figure above.

Returning now to the original subject of discussion, I assert that the ratio of a musical interval is not immediately determined either by the length, size, or tension of the strings but rather by the ratio of their frequencies, that is, by the number of pulses of air waves which strike the tympanum of the ear, causing it also to vibrate with the same frequency. This fact established, we may possibly explain why certain pairs of notes, differing in pitch produce a pleasing sensation, others a less pleasant effect, and still others a disagreeable sensation. Such an explanation would be tantamount to an explanation of the more or less perfect consonances and of dissonances. The unpleasant sensation produced by the latter arises, I think, from the discordant vibrations of two different tones which strike the ear out of time [*sproporzionatamente*]. Especially harsh is the dissonance between notes whose frequencies are incommensurable; such a case occurs when one has two strings in unison and sounds one of them open, together with a part of the other

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which bears the same ratio to its whole length as the side of a square bears to the diagonal; this yields a dissonance similar to the augmented fourth or diminished fifth [*tritono o semidiapente*].

Agreeable consonances are pairs of tones which strike the ear with a certain regularity; this regularity consists in the fact that the pulses delivered by the two tones, in the same interval of time, shall be commensurable in number, so as not to keep the ear drum in perpetual torment, bending in two different directions in order to yield to the ever-discordant impulses.

The first and most pleasing consonance is, therefore, the octave since, for every pulse given to the tympanum by the lower string, the sharp string delivers two; accordingly at every other vibration of the upper string both pulses are delivered simultaneously so that one-half the entire number of pulses are delivered in unison. But when two strings are in unison their vibrations always coincide and the effect is that of a single string; hence we do not refer to it as consonance. The fifth is also a pleasing interval since for every two vibrations of the lower string the upper one gives three, so that considering the entire number of pulses from the upper string one-third of them will strike in unison, i. e., between each pair of concordant vibrations there intervene two single vibrations; and when the interval is a fourth, three single vibrations intervene. In case the interval is a second where the ratio is 9/8 it is only every ninth vibration of the upper string which reaches the ear simultaneously with one of the lower; all the others are discordant and produce a harsh effect upon the recipient ear which interprets them as dissonances.

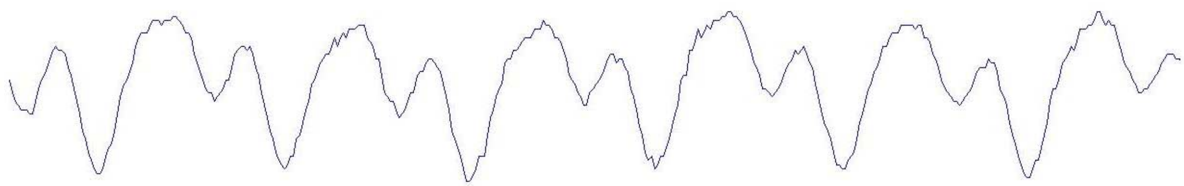
So far we have discussed three notes *do*, *sol* and the next *do*, one octave higher, whose three frequencies are in the ratio 2:3:4. Pursuing nice sounding chords leads to the whole major scale. Thus, we can add the note *mi* which has a frequency $5/4^{\text{th}}$'s above the first *do* and this gives the 'major triad' *do-mi-sol* with frequency ratios 4:5:6. Then we can go backwards creating a triad just like this but starting at the high *do*. This gives two new notes called *fa* and *la*, so that the four notes *do, fa, la, do* have frequencies in the ratio 3:4:5:6. Lastly we add a higher frequency triad, which starts at *sol*: this is *sol, ti* and *re* one higher octave. Before you get totally confused, we make a chart:

NOTE	FREQ
do	1
re	9/8
mi	5/4
fa	4/3
sol	3/2
la	5/3
ti	15/8
high *do*	2
high *re*	9/4

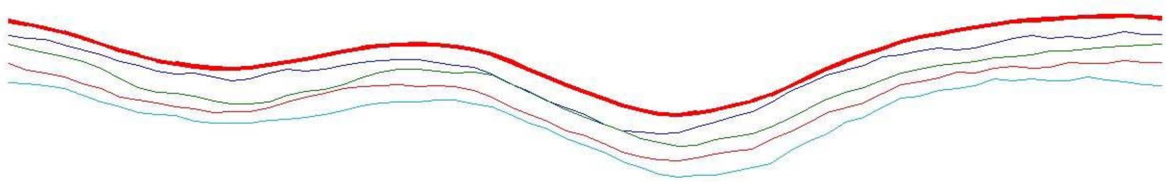
Check that *do-mi-sol*, *fa-la*-high *do* and *sol-ti*-high *re* are all major triads. With numerology like this, no wonder Pythagoras thought numbers were magic.

In fact, Galileo only guessed half the story about why these chords sound nice. We mentioned above that the curves showing the air vibration were nowhere near sinusoidal curves. However, there is a very real sense in which they are made up of a combination of basic sinusoidal curves, added together. The components are (i) the sinusoid with the same period that approximates the curve best plus (ii) a sinusoid of *double* the frequency, i.e. half the period, that makes the best correction, then (iii) a sinusoid of triple the frequency or 1/3 the period which approximates what's left, etc. These corrections are called the higher harmonics of the sound. Here's how this works:

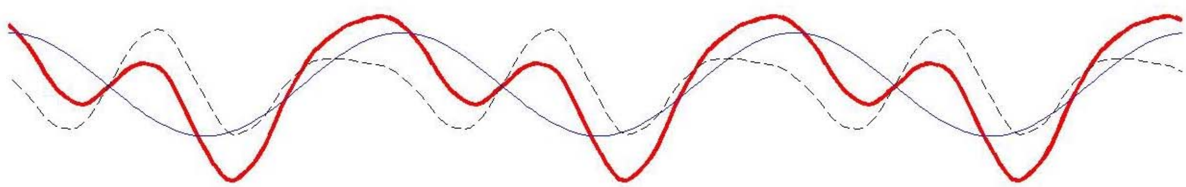
Six periods of a female voice singing the note sol



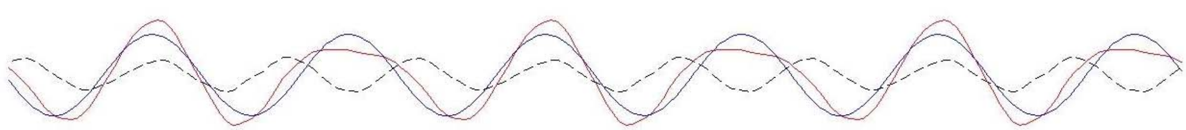
One period of the averaged detrended signal, compared to 4 samples



Three periods of a) the average signal (in red), b) its first harmonic (in blue) and c) the residual (dashed in black)



Three periods of a) the signal minus first harmonic (in red), b) its second harmonic (in blue) and c) the remaining residual (dashed in black)



On the top, you have the same voice as above singing *sol*, six periods being shown. Note that although the function has a basic period and looks like it repeats six times, there are small variations between periods. (This is less marked with a musical instrument.) On the second line, we show four examples of single periods of the voice and, in red, the average period. The average is much smoother because little tremolos in the voice have cancelled out. Then in the third line a single sinusoid has been matched to the average voice. The dashed line shows, however, the difference between the voice and its sinusoidal approximation. Remarkably, it seems to have twice the frequency. In the last graph, this residue has been approximated by a sinusoid of twice the frequency and the residue after subtracting that has been shown. The residue is very close to a sinusoid of triple the frequency. In this case, three harmonics suffice to reconstruct the voice almost exactly.

Let's put this in formulas. Let $P(t)$ be the air pressure as a function of time. Then we model this by an exactly periodic function $Q(t)$, i.e. there is a period p such that $Q(t+p) \equiv Q(t)$, all t. P and Q will be very close to each other. We write Q as a sum of sinusoids like this:

$$\boxed{Q(t) = C_0 + C_1 \sin(2\pi ft + D_1) + C_2 \sin(4\pi ft + D_2) + C_3 \sin(6\pi ft + D_3) + \cdots}$$

This is a very important formula, so we have made it big and put it in a box. The C's and D's are constants. The frequency of the whole periodic signal Q is f and the sum is made up of terms $C\sin(2\pi nft + D)$ with frequencies nf, known as the n^{th} harmonic of Q. $P(t)$ will be given by such a formula too, but, because the human voice is complicated, you have to let the C's and D's vary a bit with time. For example, in the first figure above showing *do* and *sol*, you see a slow undulation superimposed on the periodic signal: this comes from C_0 changing slowly. And if you look over longer periods of time, you find that even the shape of the signal changes slowly: this is caused by the relative phases $D_1 - D_2$ and $D_1 - D_3$ changing slowly. Another effect is vibrato, where the frequency oscillates around a mean; this is modeled by having D_1 oscillate slowly. But for the female voice used in the last figure, the change isn't too great (see second plot in the figure) and we picked a musical note for which the above three terms are already a very good approximation of the full signal $P(t)$.

Another way to say it is that hidden in the sound of *sol* is already the note *sol* one octave higher (twice the frequency) and the note *re* two octaves higher. Why *re*? From the table above, its frequency is $9/8^{\text{th}}$'s the frequency of *do*, so two octaves higher, it is $9/2^{\text{th}}$'s the frequency and $9/2 = 3 \times 3/2$, triple the frequency of *sol*! So why do chords sound well together: their harmonics overlap and they are actually sharing these hidden parts of themselves.

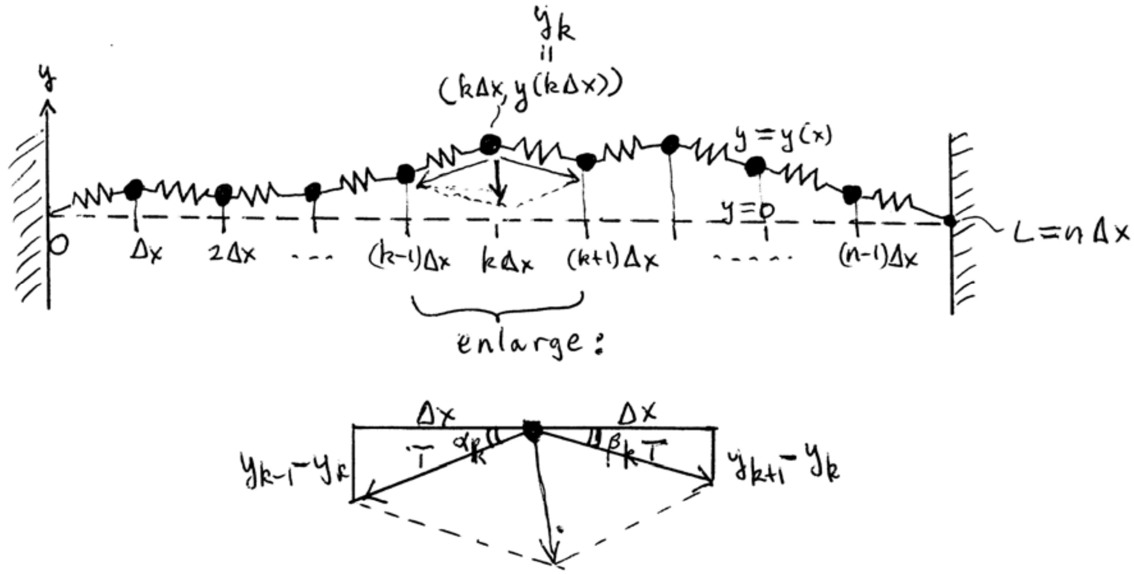
Maybe you didn't want to take a course in music theory but it's hard to resist describing the next wrinkle, namely the black keys on the piano keyboard. The major scale is the white keys and they give *do* a special place, making it a kind of home base. But composers want to play with 'changing the key' in the middle of a piece, taking another note as home and making all the triads etc on top of this. The fractions now get to be quite messy and a remarkable discovery was made: if the frequencies of the major scale are fudged a bit and 5 new notes are added (the black keys), then you get a scale in which the frequency of each note has the same ratio to the frequency of the next note, namely $2^{1/12} \approx 1.06$. Why does this work? The key piece of number magic is that $2^{7/12} = 1.498\cdots$ so a note, which is indistinguishable from sol, occurs. In fact, here are all the notes in the so-called 'tempered scale' with their frequency ratios to compared to the 'true' scale:

Note	**tempered freq. ratio**	**true freq. ratio**	**error**
C (or *do*)	1.000	1	0%
C sharp (D flat)	1.059		
D (or *re*)	1.122	9/8 = 1.125	0.2%
D sharp (or E flat)	1.189		
E (or *mi*)	1.260	5/4 = 1.25	0.8%
F (or *fa*)	1.335	4/3 = 1.333	0.1%
F sharp (or G flat)	1.414		
G (or *sol*)	1.498	3/2 = 1.5	0.1%
G sharp (or A flat)	1.587		
A (or *la*)	1.682	5/3 = 1.667	0.9%
A sharp (or B flat)	1.782		
B (or *ti*)	1.888	15/8 = 1.875	0.7%
C (one octave higher)	2.000	2	0%

Chapter Ten: The Vibrating String and PDE's

Next we want to look at the simplest physical mechanism that produces music and discover where all of this stuff with complex sound waves with many harmonics arises. The simplest musical instrument is a string (which can be made of anything – wire, gut, rubber band, ...) stretched tightly between two posts and plucked, a guitar, violin, harp, sitar, koto, etc. Newton's *F=ma* applies but with a very important new twist. This theory seems to have been first investigated by Brook Taylor (the Taylor of *Taylor series*) in a paper written in 1715, *De Motu nervi tensi* (On the motion of taut strings).

We model the string at rest as the interval on the x-axis between 0 and L and imagine it displaced a little bit in the x,y plane, so its position at time t is described by a function $y = y(x,t)$. We want to predict where the string will be at a later time $t+\Delta t$. To apply Newton's laws, we imagine the string made up of a large number of small weights, each connected to its neighbors by partially stretched springs, creating tension:



The black dots represent weights and the wiggly lines springs. We have divided up the string into (n–1) weights spaced at a distance Δx, where $n\Delta x = L$ and we have denoted the vertical displacement of each weight by $y_k = y(k\Delta x)$ and the angles by α_k, β_k. If T is the tension on the string, then, as shown in the lower enlarged diagram, there are two forces on each weight, and the force has a vertical and a horizontal component:

$$\begin{aligned}
\text{Vertical force } &= T\cdot\sin(\alpha_k) + T\cdot\sin(\beta_k) \\
&\approx T\frac{y_{k+1}-y_k}{\Delta x} + T\frac{y_{k-1}-y_k}{\Delta x} \\
&= T\cdot\frac{y_{k+1}-2y_k+y_{k-1}}{\Delta x} \\
\text{Horizontal force } &= T\cdot\cos(\alpha_k) - T\cdot\cos(\beta_k) \\
&\approx 0
\end{aligned}$$

Here the exact expression for the force involves sines and cosines, just as in the exact law for the pendulum. But just as the pendulum simplifies, when its oscillation is small, to a simple harmonic oscillator, we have also simplified the vibrating string equation by assuming the oscillation is small. This means that the two angles α_k, β_k are small, so we can replace sin(α_k) by α_k, cos(α_k) by 1 and likewise with β_k. But since cos is nearly 1, we also have $\sin(\alpha_k) \approx \tan(\alpha_k) = (y_{k-1} - y_k)/\Delta x$. These are the simplifications we made above.

Now apply Newton's law $F=ma$. The vertical force is causing an acceleration of the k^{th} mass. If d is the density of the string per unit length, then the k^{th} mass should equal $d \cdot \Delta x$, so we get (writing $y(x,t)$ instead of $y_k(t)$):

$$T \cdot \frac{y((k+1)\Delta x, t) - 2y(k\Delta x, t) + y((k-1)\Delta x, t)}{\Delta x} \approx \text{ force}$$

$$= \text{mass } \times \text{ accel. } \approx (d\Delta x) \times \left(\frac{y(k\Delta x, t+\Delta t) - 2y(k\Delta x, t) + y(k\Delta x, t-\Delta t)}{\Delta t^2} \right)$$

Now $y(x,t)$ describes the string's motion is a function of both space and time and so, as Oresme saw early on, it can be uniform, difform, etc in x and in t, and these mean different things. In other words, we can hold t fixed and consider the rate of change and the rate of change of the rate of change when x varies; or we can hold x fixed and do the same for t. The first gives the first and second derivatives of y with respect to changes in x alone; and the second gives derivatives with respect to t alone. Clearly the above expression is approximating the second derivatives of y with respect to x on the left and with respect to t on the right. The first derivatives are usually written as y_x or $\partial y/\partial x$ for the derivative when x is varied; y_t or $\partial y/\partial t$ when t is varied. And the second derivatives are written y_{xx} or $\partial^2 y/\partial x^2$ when x is varied, y_{tt} or $\partial^2 y/\partial t^2$ when t is varied. So divide by Δx, let Δx and Δt go to zero and the equation above comes out as what is now called a *partial differential equation* (because both space and time derivatives enter):

$$\boxed{T \cdot y_{xx} = d \cdot y_{tt}}$$

This is called the vibrating string equation or the one-dimensional wave equation. Note that in its discrete form, we can solve it for $y(k\Delta x, t+\Delta t)$ in terms of the position of the string at time t and at time $t - \Delta t$. Thus we have a rule for predicting the future, exactly similar to the rules used before for predicting simple springs, pendulums, foxes and rabbits, etc. Essentially, what we have done is to describe the state of the universe at t=0 not by 1 or 2 or any finite set of numbers, but by a whole function $y(x,0)$ and then to give a rule by which this function in the future can be found. Actually, we must be a bit careful: you need to know $y(x,0)$ *and* $y_t(x,0)$ to solve the wave equation forward in time. You can see this from the discrete version, which requires y at times t and $t - \Delta t$ in order to move forward.

This equation was the subject of a great deal of study – and controversy – in the eighteenth century. It started with the short note by Taylor mentioned above who stated (geometrically but essentially equivalent to the above!) that *the normal acceleration of the string was proportional to its curvature*. For some 50 years after this, Daniel Bernoulli, Leonard Euler and Jean D'Alembert wrote and argued about this equation back and forth in the pages of the learned journals of the day, with many confusions and gradual incremental progress. They were all leading figures, especially Euler, but, as we shall see, this argument became the source of Euler's biggest error in his otherwise amazing career.

We saw how a musical voice was not a simple sinusoidal vibration in the previous Chapter. Now that we have an equation for how all string instruments produce vibrations, let's see if we can understand the mysteries of music and integers more deeply. Let's start by cooking up a function $y(x,t)$, which satisfies the wave equation out of a few trig functions. The string is fixed at each end, so if its ends are $x=0$ and $x=L$, we need to have $y(0,t)=0$ and $y(L,t)=0$ for all t. What's a good starting position for the string? If it has one simple bow, why not try $\sin(\pi x/L)$. This is indeed zero at both $x=0$ and $x=L$ and between them, makes a simple positive arc. If the string is going to vibrate up and down, what could be simpler than multiplying this by some function of time that starts at 1 and oscillates at frequency f between +1 and -1? This suggests that:

$$y(x,t) = \sin(\pi x / L) \cdot \cos(2\pi ft)$$

has all the right properties. Taking its derivatives, we find:

$$\begin{aligned} y_x &= (\pi / L)\cos(\pi x / L) \cdot \cos(2\pi ft) \\ y_{xx} &= -(\pi / L)^2 \sin(\pi x / L) \cdot \cos(2\pi ft) \\ y_t &= -2\pi f \sin(\pi x / L) \cdot \sin(2\pi ft) \\ y_{tt} &= -(2\pi f)^2 \sin(\pi x / L) \cdot \cos(2\pi ft) \end{aligned}$$

so this satisfies the wave equation, provided that the frequency is chosen correctly:

$$f = \frac{1}{2L}\sqrt{\frac{T}{d}}$$

(Just plug everything into the wave equation and check.) Right off, we see the first demonstration of musical theory: halve the length of the string and frequency doubles, we have the octave. Divide the string in four and we have a frequency four times higher, the octave of the octave, etc. But this also shows how the frequency depends on the weight and tension of the string, relations that Galileo discusses in his dialogues.

This is all very nice, but we can look for more solutions y like this. As Daniel Bernoulli noted in 1728, one can also insert a little integer n into the formula:

$$y(x,t) = \sin(\pi nx / L) \cdot \cos(2\pi nft + D)$$

and it still satisfies the wave equation (just note that both first derivatives are increased by n and both second derivatives by n^2). We also inserted a phase D into the time dependence, which doesn't change anything (we can't do this with x because the ends of the string have to remain fixed). Now this wave has n times the frequency and causes the string to vibrate, not with one arc moving up and down, but with n arcs moving alternately up and down. For $n > 1$, these are called the higher modes of vibration of the string, higher both in frequency and the complexity of their shape. Bernoulli was an applied mathematician (he worked extensively on hydrodynamics and elasticity) and his paper is notable in that he followed exactly the derivation of the vibrating string equation that we have just given: his title was *Meditations on vibrating strings with little weights at equal distances.* He then passed to the limit and found the sine curve – still called at that point "the companion of the cycloid"! In a later 1753 paper, he attributes these solutions to Taylor and makes some sketches:

II. Let us first observe that, according to Mr. Taylor's theory, a stretched string can perform uniform vibrations in an infinity of ways, physically speaking different from each other, but geometrically speaking amounting to the same, since in every one of them only the unit that serves as measure is changed. These different ways are characterized by the number of loops [*ventres*] that the string can form during its vibration. When there is only one loop [Fig. 1], then the vibrations are the slowest, and they produce the fundamental tone; when there are two loops, and one node [*noeud*] in the middle of the axis [Fig. 2], then the vibrations are doubled, and they produce the octave of the fundamental tone; when the string forms three, four, or five loops, with two, three, or four nodes, at equal distances, as in Figs. 3, 4, 5, then the vibrations are multiplied by three,

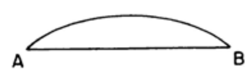


Fig. 1

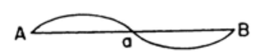


Fig. 2

Fig. 3

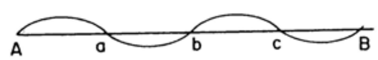


Fig. 4

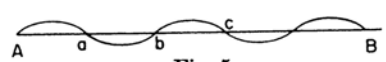


Fig. 5

four, or five, and produce the twelfth, the double octave, or the major third of the double octave relative to the fundamental tone. In every type of these vibrations the total displacements can be large, or small, at discretion, provided that the largest must be considered as extremely small. The nature of these vibrations is such that not only does each point begin and end every simple vibration at the same instant, but also all the points place themselves after every simple half-vibration in the position of the axis AB. We must regard all these conditions as essential, and then we have at once the curves described by Mr. Taylor as satisfying the problem.

He goes on to say:

> *My conclusion is that all sounding bodies contain potentially an infinity of tones, and an infinity of corresponding ways of performing their regular vibrations – in short, that in every different kind of vibration the inflections of the parts of the sounding body are made in a different way.*

But the story is not finished. There are yet more solutions because *the equation is linear*! This means that where y appears, it is multiplied by stuff or differentiated but it is never squared or put into a non-linear function like sine. So if we have two solutions of the equation, we can add or subtract them or multiply them by constants making them bigger or smaller and any such operation gives us more solutions. In other words, all functions:

$$y(x,t) = C_1 \sin(\pi x / L) \cdot \cos(2\pi ft + D_1) + C_2 \sin(2\pi x / L) \cdot \cos(4\pi ft + D_2) + C_3 \sin(3\pi x / L) \cdot \cos(6\pi ft + D_3) + \cdots$$

are solutions of the wave equation. This (potentially infinite) sum seems to have been implied in a paper of Bernoulli in 1741, where he states that the various modes of oscillation can exist together. But it was written down first by Euler in 1749, who referred to the shape of the resulting function as a *courbe anguiforme*, an "eel-like curve". Note that if we freeze x and consider this as a function of t, we have exactly the expression used in the last Chapter to model the singing voice. We see that the wave equation has given us the key to explain all the complexities, which had been hidden in music for so long. It has united three of the four parts of the quadrivium – arithmetic, geometry and music.

Euler's paper apparently took Bernoulli by surprise because Bernoulli had not written out his full theory very explicitly. Moreover, as we shall discuss in Chapter 13, D'Alembert and Euler had found a completely different way to express solutions, which were not obviously reducible to the formula above. In the polite discourse of the Enlightenment, he wrote in the same 1753 paper quoted above, that these new solutions were "improper" though strictly speaking correct solutions!

> I. Mr. Taylor was the first to obtain the number of vibrations made in a given time by a string uniformly thick, of given length and given weight, and stretched by a given force. It was not possible to determine this number without knowing in advance the curve taken by the string during the whole time that its vibration lasted; he therefore proved that this curve was always "the companion of an extremely elongated cycloid," for which the ordinates represent the sines of the arcs represented by the abscissas. I think that only in this form can the vibrations become regular, simple, and isochronous despite the inequality of the deviations [*excursions*]. Since I always had this idea I could only be surprised to see in the *Mémoires* [of the Berlin Academy] of the years 1747 and 1748 an infinity of other curves claimed to be endowed with the same property. I really needed the great names of Messrs. D'Alembert and Euler, whom I could not suspect of any carelessness, to make me examine whether there would not be anything in this aggregate of curves that conflicted [*équivoque*] with those of Mr. Taylor, and in what sense they could be admitted. I immediately saw that this multitude of curves could be admitted only in quite an improper sense. I do not the less esteem the calculations of Messrs. D'Alembert and Euler, which certainly contain all that analysis can have at its deepest and most sublime, but which show at the same time that an abstract analysis which is accepted without any synthetic examination of the question under discussion is liable to surprise rather than enlighten us. It seems to me that we have only to pay attention to the nature of the simple vibrations of the strings to foresee without any calculation all that these two great geometers have found by the most thorny and abstract calculations that the analytical mind can perform.

If we look at a simple example, we will be able to see perhaps part of what Bernoulli meant when he called D'Alembert and Euler's new solutions "improper". Let's start with the string stretched along a curve:

$$y(x,0) = \sin(\pi x / L) + 0.5 \cdot \sin(2\pi x / L)$$

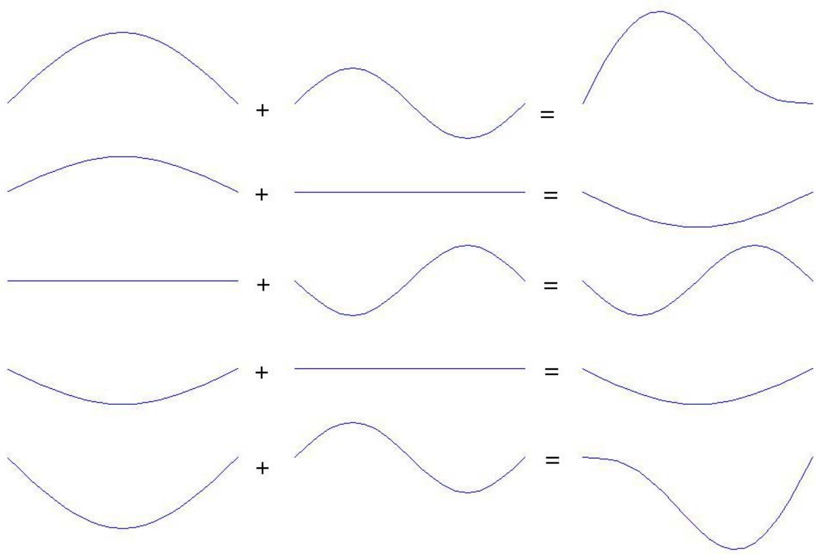
and with no velocity, $y_t(x,0) = 0$ and then let it go. What will the string do? Our general formula above tells us that the solution has the form:

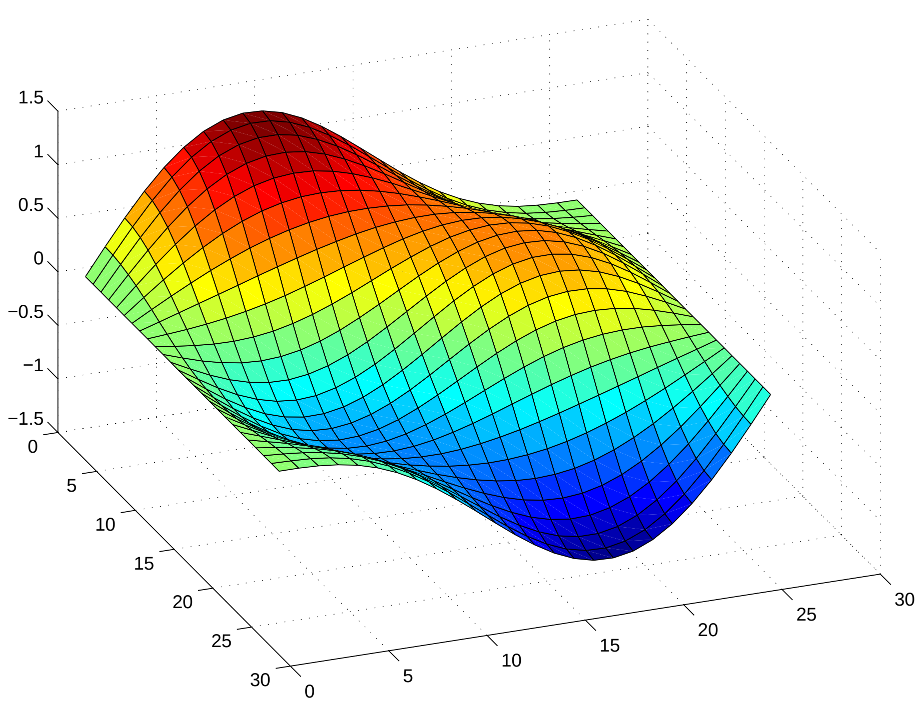
$$y(x,t) = C_1 \sin(\pi x / L)\cos(2\pi ft + D_1) + C_2 \sin(2\pi x / L)\cos(4\pi ft + D_2)$$

and we have to fit the coefficients C_1,C_2,D_1 and D_2. This is easy: to make the t-derivative y_t zero at t=0, we need only set D_1=D_2=0. To make y start at the right place, we set C_1 = 1 and C_2 = 0.5. But what does this look like? Let p = 1/f be the period of the first term (so p/2 is the period of the second term). It's easy to graph

$$y(x,t) = \sin(\pi x / L)\cos(2\pi ft) + 0.5 \cdot \sin(2\pi x / L)\cos(4\pi ft)$$

The figure on the left shows the 2 terms and their sum for t equal to 0, p/8, p/4, 3p/8 and p/2. Note how the first term makes half a cycle while the second term has both twice the spatial and twice the time frequency, hence makes a complete cycle. A sexier way to display this function of two variables is to use the colored 3D 'mesh' plot shown on the left.





Chapter Eleven: Fourier Series and Spectrograms

We have skirted around two obvious questions:

- Is every solution of the vibrating string equation of the form written down by Euler:
  $$y(x,t) = C_1 \sin(\pi x / L) \cdot \cos(2\pi ft + D_1) + C_2 \sin(2\pi x / L) \cdot \cos(4\pi ft + D_2) + \\ + C_3 \sin(3\pi x / L) \cdot \cos(6\pi ft + D_3) + \cdots$$
- Is every function $y(x)$ with $y(0) = y(L) = 0$ a sum of sinusoids like this:
  $$y(x) = C_1 \sin(\pi x / L) + C_2 \sin(2\pi x / L) + C_3 \sin(3\pi x / L) + \cdots$$

The answer is YES to both IF you're a little careful about how jumpy and erratic the functions y are allowed to be and about how to add up an infinity of terms of higher and higher frequency. (These 'IF's are the typical questions that can occupy months of study in higher math courses but that are usually irrelevant for applications and computations.)

Both of these follow from a third fact, which was one of the most important mathematical discoveries of the 18th and 19th century:

- Every periodic function $y(x)$ satisfying $y(x+p) = y(x)$ for all x and some fixed
- period p can be expanded into a sum of both sines and cosines:
  $$y(x) = B_0 + A_1 \sin(2\pi x / p) + B_1 \cos(2\pi x / p) + A_2 \sin(4\pi x / p) + B_2 \cos(4\pi x / p) + \cdots$$
  or (using the rule that $A\sin(x) + B\cos(x)$ can be rewritten as $C\sin(x+D)$), into a sum of sines with phase shifts:
  $$y(x) = C_0 + C_1 \sin(2\pi x / p + D_1) + C_2 \sin(4\pi x / p + D_2) + \cdots$$

The 'take home message' is that writing a function as a sum of sines and cosines is almost as important as writing a function as a polynomial: they are universal tools that display basic parameters in the functions makeup.

Relationship between the three bullets:

(a) We get sine series as a special case of Fourier series for this reason: if $y(x)$ is defined between 0 and L and is zero at 0 and L, then first extend y to a function between $-L$ and 0 to have values $-y(-x)$ and then make y into a function defined for all values of x by making it periodic with period $2L$. Then the Fourier expansion of this periodic extension y turns out only to have sine's in it because y is 'odd', $y(-x)=-y(x)$ for all x, and so we get the sine series.

(b) If we know that Fourier and sine series always exist, then we also know that these series give all the solutions of the vibrating string equation – the first bullet. This is because we saw from the difference equation approach to the PDE that if we know where y starts and its rate of change, i.e. $y(x,0)$ and $y_t(x,0)$, then the equation has only one solution. So we just need to write out the two sine series:

$$y(x,0) = B_1 \sin(\pi x / L) + B_2 \sin(4\pi x / L) + \cdots$$
$$y_t(x,0) = 2\pi \left(A_1 \sin(\pi x / L) + 2A_2 \sin(4\pi x / L) + \cdots \right)$$

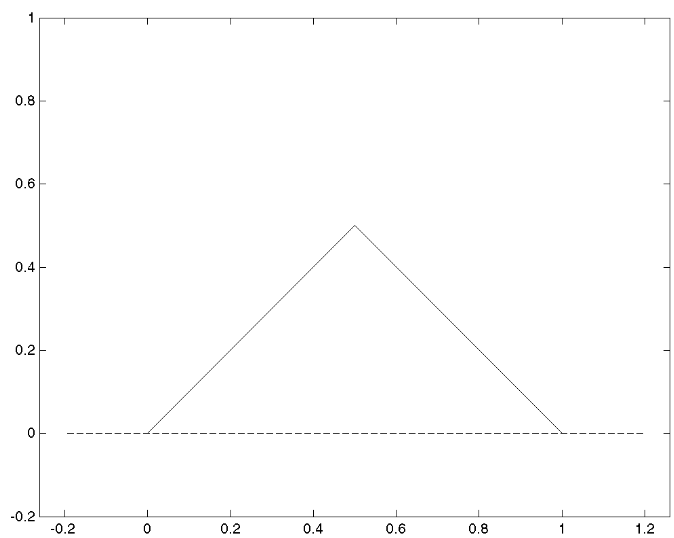
and contrive the C's and D's so that

$$A_k \sin(2\pi kft) + B_k \cos(2\pi kft) = C_k \cos(2\pi kft + D_k)$$

With this choice of C's and D's, the function in bullet 1 is easily seen to have the right values of $y(x,0)$ and $y_t(x,0)$.

The theorem that all, not too erratic, periodic functions $y(x)$ have Fourier expansions has one of the most curious histories one could imagine. Euler, who found and published such expansions for all the basic functions and who loved manipulations of this kind, resisted strongly the idea that *every* function could be so expanded. The great mathematicians of the 18th century were polarized: on one side, the mathematicians who were mostly 'pure' mathematicians, Euler, D'Alembert and Lagrange, insisted that functions given by Fourier series were special and, on the other side, the truly applied mathematicians and mathematical astronomers, Bernoulli and Clairaut, believed it was true. Fourier, after whom these series are named, was a 19th century polymath, who split his career between teaching and serving as prefect in various Departments of France, and applied these series to understand the spread of heat in the earth. Although not the inventor of 'Fourier series', he claimed strongly that they did represent all periodic functions and stimulated the rigorous theory of these series whose twists have continued to this day[1].

The best way to understand what was at issue is to look at an example. Euler had the idea that a sum of trig functions could be made to add up to any $y(x)$ that could be given by a single closed formula, such as a polynomial. But he also introduced what was then a radically new idea of what a function was: it could be given by one formula for some values of their argument and another function for others (or it might even be a freehand curve, drawn by hand). He called these *discontinuous* because the formula for them changed abruptly even though their value need not jump or anything. A typical example is the tent function:



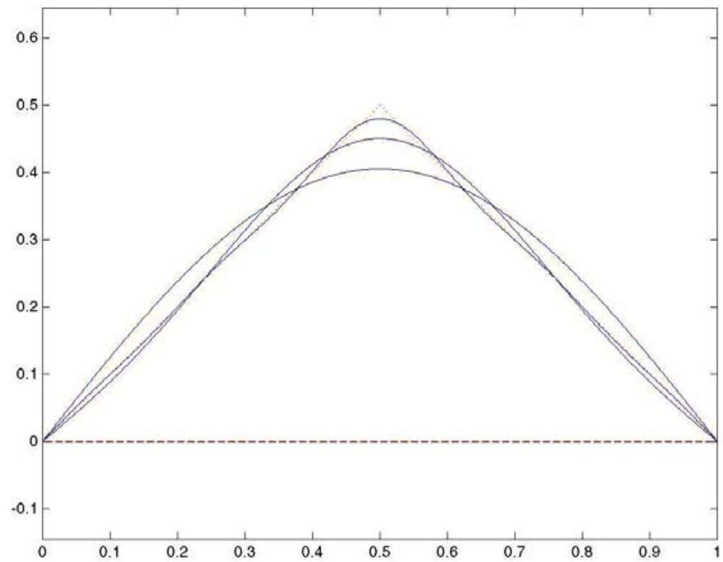
$$y(x) = |x|, \quad \text{if } 0 \le x \le L/2$$
$$y(x) = |L - x|, \text{ if } L/2 \le x \le L$$

which is shown on the right for L = 1. Euler felt he could expand the first part and the second half into trig functions but not the combination of the two. *But he was wrong.* The answer is this (with $L = 1$ for simplicity):

$$y(x) = C\left(\sin(\pi x) - \frac{1}{9}\sin(3\pi x) + \frac{1}{25}\sin(5\pi x) - \cdots\right), \text{ where } C = 4/\pi^2$$

[1] The key issues are how erratic a function can be to be expanded in such a series and in what sense this infinite series converges for increasingly wild functions.

This is not irrelevant to music: the above shape is a plausible way to pluck a string and the expansion shows that it produces all *odd* harmonics, that is the note itself, then its third harmonic, then its fifth, etc. Let's graph this expression truncating the infinite sum of sines – thus check it numerically, and then we'll see how to find such facts.



Here is a plot of three approximations to the tent curve, (a) with a single sine, (b) with two sines $\sin(\pi x) - \sin(3\pi x)/9$ and (c) with five sine terms of frequencies 1,3,5,7 and 9. It gets close to the tent everywhere except at the peak, but eventually, the trig sum will get close at the peak too (though each finite sum will be round at the peak of the tent if you look closely).

Where did we get these strange coefficients 1/9, 1/25 etc and outside everything $4/\pi^2$? There's a simple trick that can be used! If we want the coefficient of the first term $\sin(\pi x)$, we multiply both sides of the equation by $\sin(\pi x)$, so that the equation reads:

$$\sin(\pi x)\cdot y(x) = C\left(\sin^2(\pi x) - \frac{1}{9}\sin(\pi x)\cdot\sin(3\pi x) + \frac{1}{25}\sin(\pi x)\cdot\sin(5\pi x) - \cdots\right)$$

The point is that $\sin^2(\pi x)$ is always positive (or 0) while the other terms on the right are both positive and negative. In fact, the average value of $\sin^2(\pi x)$ is ½ and the average value of the others is 0. So just integrate both sides of the equation between 0 and 1 and we'll get an equation that can be solved for C!

Let's check the details here:

a) We use the trig identity $\cos(u+v) = \cos(u)\cos(v) - \sin(u)\sin(v)$. Replacing v by $-v$, this also gives $\cos(u-v) = \cos(u)\cos(v) + \sin(u)\sin(v)$ and subtracting these two, we get $\sin(u)\sin(v) = ½\,(\cos(u-v) - \cos(u+v))$. Let $u=v=\pi x$, and we get $\sin^2(\pi x) = \big(1-\cos(2\pi x)\big)/2$. Now any expression $\sin(2\pi x)$ or $\cos(2\pi x)$ has symmetrical positive and negative lobes hence has average 0 between 0 and 1. So $\sin^2(\pi x)$ has average ½.

b) We use the same identity to rewrite $\sin(n\pi x).\sin(m\pi x)$ for $n \neq m$. It shows it equals ½ $(\cos((n-m)\pi x) - \cos((n+m)\pi x))$ and now both cosine terms have average 0.

c) We conclude $\int_0^1 y(x)\sin(\pi x)dx = C\int_0^1 \sin^2(\pi x)dx = C/2$.

d) Now the integral on the left is easy to compute *if* you remember a trick from calculus – integration by parts. It works out like this (but just accept this if you want):

$$\int_0^1 y(x)\sin(\pi x)dx = 2\int_0^{1/2} x\sin(\pi x)dx$$
$$= \frac{2}{\pi}\left(-x\cos(\pi x)\Big|_0^{1/2} + \int_0^{1/2}\cos(\pi x)dx\right) = \frac{2}{\pi}\left(0 + \frac{\sin(\pi x)}{\pi}\Big|_0^{1/2}\right) = \frac{2}{\pi^2}$$

e) Thus $C = 4/\pi^2$!

f) This trick works to get all the coefficients. Using (a) and (b), we first show that the coefficient of $\sin(n\pi x)$ is equal to $2\int y(x)\sin(n\pi x)dx$. This is easily worked out by integration by parts as in (d) and that it equals 0 if n is even, and equals $(2/n\pi)^2$ if n is odd.

In fact, we can expand any periodic function into a sum of sines and cosines, finding its coefficients by the above trick, known as the *orthogonality* of the trig functions, the fact that the average value of products of different trig functions is zero. We give another example in the problem below.

Let's formulate the general rule. Suppose $y(x)$ is now any periodic function, with period p. That is, $y(x+p) = y(x)$ for all x. Then $y(x)$ can be written uniquely as an infinite sum of trig terms:

$$y(x) = B_0 + A_1\sin(2\pi\frac{x}{p}) + B_1\cos(2\pi\frac{x}{p}) + A_2\sin(4\pi\frac{x}{p}) + B_2\cos(4\pi\frac{x}{p}) + \cdots\cdots$$

Moreover, the coefficients can be found by:

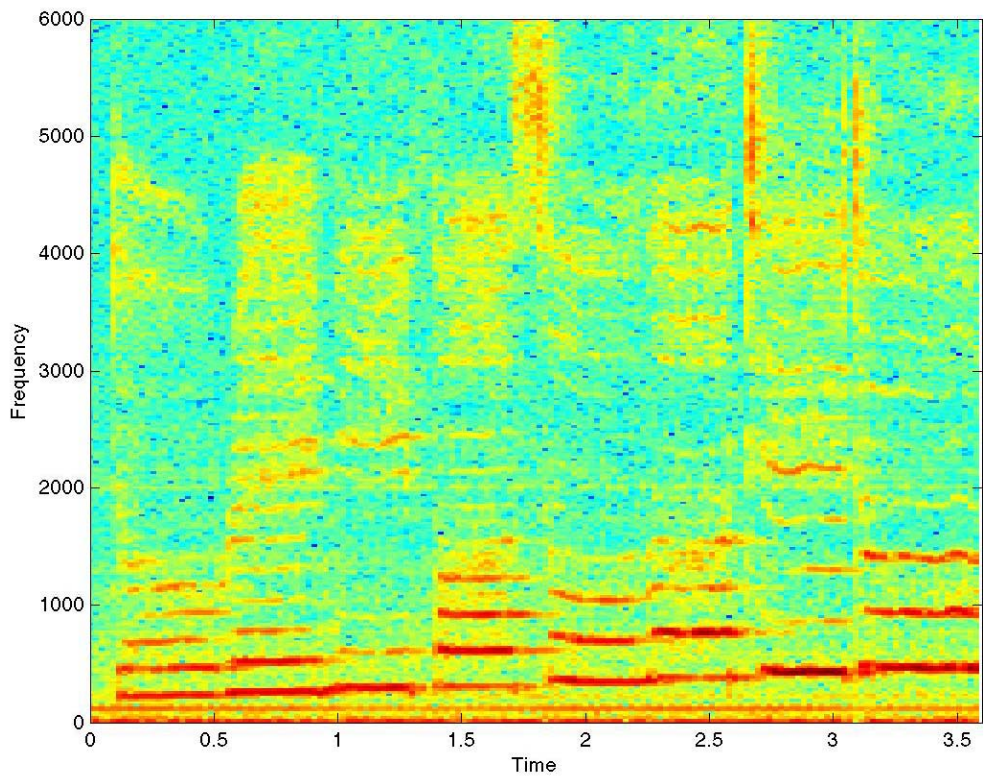
$$A_k = \frac{2}{p}\int_0^p \sin(2k\pi\frac{x}{p})y(x)dx,$$
$$B_0 = \frac{1}{p}\int_0^p y(x)dx$$
$$B_k = \frac{2}{p}\int_0^p \cos(2k\pi\frac{x}{p})y(x)dx, \text{ if } k > 0$$

This called the Fourier Series for $y(x)$. Of course, each sine/cosine pair can be rewritten:

$$A_k\sin(2k\pi\frac{x}{p}) + B_k\cos(2k\pi\frac{x}{p}) = C_k\left(\sin(2k\pi\frac{x}{p} + D_k)\right)$$

where $C_k = \sqrt{A_k^2 + B_k^2}$ is called the amplitude of the k^{th} harmonic and D_k its phase (C_k^2 is called its *power*).

There's a beautiful graphical way to display music using Fourier series that is the mathematical version of a musical score! It's called a *spectrogram.* All you need to do is break the sound up into 'windows' and expand each piece[2] into a Fourier series as above and make a picture out of the amplitude of the various harmonics in each window. In the figure below, the amplitude of the coefficients has been graphed by *colors* (another twist on Oresme's precepts!):



The spectrogram of the major scale sung by a female voice. The 8 vertical strips show the 8 notes *do, re, mi, fa, sol, la, ti* and *do.* The dark horizontal lines in each strip represent strong harmonics present in each note: the lowest is the fundamental, then the second harmonic etc. *Ti* for example shows only the fundamental and the fifth harmonic and also has a marked trill. *La s*hows significant power in the 11th harmonic. Note the dark high frequency signal between *fa* and *sol*: this is called 'white noise' and what happens when you speak an 's'. It also occurs as a burst in the stop consonants 't' and 'd' of *ti* and the final *do*.

[2] What you do exactly is take a 'window function', a smooth hump-like function (a) zero outside the window, (b) 1 on 90% of the window and (c) with shoulders near the beginning and end of the window. You multiply the sound by this window function and treat the product as though it were a periodic function, wrapping the beginning and end of the window together. Then it expands as a Fourier series by the formula above.

Chapter Twelve: Trigonometry and Imaginary Numbers

Although we could skip this if we were trying to explain, in simplest terms, the mathematical models of waves with differential equations, the story of the square root of minus one is such an amazing one, such an unexpected twist, that it begs to be included. Of all the devices that mathematicians have found and used to model nature, this one seems – to me at least – as if God decided to throw us a curve ball, something unexpected, something that 'needn't have been so' but nonetheless really is true. My Aunt, studying maths at Girton College nearly a hundred years ago, called the square root of minus one a 'delightful fiction'.

Episode I: Over millennia, many cultures have wanted to solve polynomial equations. Along with trying to make sense of the planets and the moon, solving quadratic equations appears as a strange mathematical obsession with so many cultures. We saw that the Babylonians got into this in a big way and the Chinese, Greeks, Indians and Arabs all solved the quadratic equations.

There is always an issue when you do this: solved quadratic equations with what sort of numbers? Usually, this meant positive real numbers as negative numbers were not legitimate. But both the Chinese and the Indians introduced negative numbers, e.g. to represent debts. The 12th century Indian mathematician Bhaskara addressed the issue of square roots explicitly: he states that positive numbers have 2 square roots, the usual positive one and its negative – but that negative numbers have no square roots. He was well acquainted with the fact that the product of a negative and positive number is negative while the product of 2 negative numbers is positive. Incidentally, he also said when he found an equation with one positive and one negative solution: "*The (negative) value is in this case not to be taken, for it is inadequate; people do not approve of negative solutions*".

In the late Renaissance (c.1500-1550), it became a sporting competition for Italian mathematicians to challenge each other to solve various higher degree equations. Ferro, Tartaglia, Cardano and Ferrari were players. But Girolamo Cardano (1501-1576) spoiled the game by publishing their secrets in his famous book *Ars Magna* (The Great Art) in which the general procedure for solving third and fourth degree equations was explained. He was a boisterous figure, interested in everything, a gambler who was jailed for casting the horoscope of Jesus. But in his book, negative numbers were treated with suspicion and called *fictitious* solutions. Fourth powers were likewise a game – squares stood for areas of squares, cubes stood for volumes of cubes, but, in the absence of a fourth dimension, what should one make of fourth powers?

These sound like silly scruples to us now. But once you play games with formulas to solve equations, not only do you find it much easier to allow negative numbers and fourth powers, but you find it hard not to take square roots of all numbers. Cardano, in particular, found a very strange thing: his formula for the solutions of cubic equations, which usually worked fine, sometimes involved intermediate steps which were square roots of negative numbers, even when the final answer should be an honest positive real number! He ends his book saying "*So progresses arithmetic subtlety, the end of which, as is said, is as refined as it is useless*". His point of view was the same as my Aunt's.

Episode II: Skip ahead 200 years to the Enlightenment. At this point, mathematicians were much more familiar with square roots of negative numbers and had introduced the general class of numbers, called *complex* numbers of the form:

$$a+\sqrt{-1}\cdot b$$

where a and b were arbitrary real numbers. (Among these, real numbers are those with b=0 and imaginary numbers are those with a=0.) It was not hard to verify that you can (i) add two such – by adding the a's and b's; (ii) multiply two such by following the usual rules and using one new one $\sqrt{-1}\times\sqrt{-1}=-1$; (iii) that then we can also subtract and divide by any non-zero number. And D'Alembert had shown that really remarkable fact that no further fictitious or imaginary numbers need to be invented in order to solve further polynomial equations: every n^{th} degree equation has n roots if you allow them to be complex numbers. This was called the fundamental theorem of algebra, a reasonable name since it capped more than 3 millennia of solving polynomial equations. In spite of all this, there was an air of unreality about them. Felix Klein writes about this period like this:

> "*Imaginary numbers made their own way into arithmetic calculations without the approval, and even against the desires of individual mathematicians, and obtained wider circulation only gradually and to the extent that they showed themselves useful*".

Euler, who loved formulas with all his heart, founds a link between trigonometry and the square root of minus one and comes up with the weirdest formula of all, or perhaps it should be called the formula which finally made it clear imaginary numbers were useful. We have repeatedly made use of the somewhat cumbersome addition formulas for sine and cosine:

$$\sin(x+y)=\sin(x)\cos(y)+\cos(x)\sin(y),$$
$$\cos(x+y)=\cos(x)\cos(y)-\sin(x)\sin(y).$$

Well, let's look at the magical combination which we temporarily name $e(x)$

$$e(x)=\cos(x)+\sqrt{-1}\cdot\sin(x)$$

Now do one simple thing: write out $e(x+y)$ and $e(y)$ using the trig formulas:

$$\begin{aligned} e(x+y) &= \cos(x+y)+\sqrt{-1}\sin(x+y) \\ &= \cos(x)\cos(y)-\sin(x)\sin(y)+\sqrt{-1}\sin(x)\cos(y)+\sqrt{-1}\cos(x)\sin(y) \\ &= \left(\cos(x)+\sqrt{-1}\sin(x)\right)\left(\cos(y)+\sqrt{-1}\sin(y)\right) \\ &= e(x)e(y) \end{aligned}$$

Now isn't that simpler! In particular, it has as a Corollary a formula that DeMoivre had found a few years earlier. If x=y, then we get $e(2x)$=$e(x)^2$; and so if x=2y, we get $e(3x)$=$e(x).e(2x)$=$e(x)^3$; and proceeding by induction, we get:

$$e(nx)=e(x)^n, \text{ or}$$
$$\cos(nx)+\sqrt{-1}\sin(nx)=\left(\cos(x)+\sqrt{-1}\sin(x)\right)^n$$

But it was Euler who made the next wonderful leap: fix the product $y=nx$ but let n get larger and larger, x get smaller and smaller. Then $\sin(x)$ is very close to x and $\cos(x)$ is very close to 1, so:

$$\begin{aligned} e(y) &= \cos(y)+\sqrt{-1}\sin(y) \\ &= \left(\cos(y/n)+\sqrt{-1}\sin(y/n)\right)^n \\ &\approx \left(1+\sqrt{-1}y/n\right)^n \end{aligned}$$

Now remember the rule for compound interest, for continuous exponential growth. This was that:

$$e^{ax} = \lim_{n\to\infty}\left(1+a/n\right)^n$$

So now Euler makes the conclusion $e(y)=e^{\sqrt{-1}y}$, or, written out:

$$\boxed{\cos(y)+\sqrt{-1}\sin(y)=e^{\sqrt{-1}y}}$$

What in heaven's name does this mean? Well, nothing really; or better, it is really a definition that one is compelled to make in order to keep arithmetic working smoothly. Its most astonishing corollary is the special case when $y=\pi$, when it says:

$$e^{\sqrt{-1}\pi}=-1$$

I once had to give an after dinner talk to a distinguished group of non-mathematicians about mathematics, and thought, can I explain this weird formula to them? Here's the explanation:

> Suppose an imaginative and enterprising banker decides to offer an exciting new type of savings account – one that pays imaginary interest, at the rate of $(10\sqrt{-1})$% each year. The public, fascinated by imaginary money, wants to participate in this new financial offering. Joe Bloggs deposits \$100 in such an account. After one year, he has earned 10 imaginary dollars in interest and his balance stands at \$$(100+10\sqrt{-1})$. The next year he gets 10 more imaginary dollars and is thrilled: but to his chagrin, the imaginary balance of 10 imaginary dollars also earns interest of $(0.1\sqrt{-1})$ x \$$10\sqrt{-1}$ dollars, or –1 real dollars. So his balance after two years stands at \$$(99+20\sqrt{-1})$. As the years go by, he keeps building up his pile of imaginary dollars but, as this gets bigger, he also sees the interest on this imaginary balance whittle away his real dollars at an ever-increasing rate. In fact, if the bank used continuous compounding of interest rather than adding the interest once a year, then after 5 years Joe would have \$$(88+48\sqrt{-1})$, having lost 12 real dollars in return for his 48 imaginary ones. Joe doesn't quite know what these imaginary dollars are good for, but maybe they aren't a bad deal in return for the 12 real ones he lost! Time passes and, after 10 years, he has \$$(54+84\sqrt{-1})$ and now his real money is bleeding away fast because of the interest on his imaginary balance. In fact, at 15 years, his balance is \$$(7+99.5\sqrt{-1})$ and finally at 15 years, 8 months and 15 days he checks his balance, only to find he no real money at all, but 100 imaginary dollars. This length of time is in fact 10π/2 years and what we

have done is track his balance by Euler's formula. Explicitly, since continuous compounding is the same as using exponentiation, we have:

$$\begin{aligned} \text{Balance after } t \text{ years } &= (\text{Initial deposit}) \times \mathrm{e}^{(\text{Interest rate})t} \\ &= \$100 \times e^{0.1\sqrt{-1}t} \\ &= \$100\left(\cos(0.1t) + \sqrt{-1}\sin(0.1t)\right) \end{aligned}$$

Let's go on. More years elapse and now the interest on Joe's imaginary dollars puts him in real debt. *And* the interest on the real debt begins to take away his imaginary dollars. At 20 years, his balance stands at $\$(-41+91\sqrt{-1})$, at 25 years $\$(-80+60\sqrt{-1})$ and at 30 years $\$(-99+14\sqrt{-1})$. Finally at 10π years, which works out to be 31 years, 5 months, he finds himself 100 dollars in debt with no imaginary money. Not willing to give up, and finding the banker willing to extend him credit with only imaginary interest to pay, he perseveres and after about 47 years, finds that he has only imaginary debt now, and no real money either positive or negative. And now the interest on negative amounts of imaginary money is positive real money (because $(0.1\sqrt{-1})$ x $(-100\sqrt{-1}) = +10$). So he finally begins to win back his real money. On his deathbed, after 20π years, that is 62 years and 10 months, he has back his original deposit and has paid off his imaginary debt. He promptly withdraws this sum, sues his banker and vows never to have any truck with complex numbers again. His odyssey is traced in the figure below.

Episode III: A Norwegian surveyor invents the complex plane and, finally, Gauss makes it all seem respectable, even mundane.

Episode IV: Quantum mechanics makes the state of the world into a complex superposition, thus embedding the square root of minus one deeply in God's plan for the universe.

Chapter Thirteen: Traveling Waves in 1 and more dimensions

First, the traveling wave solution of the vibrating string equation. On the infinite line, on the finite line, 'reflecting' off the endpoints.

Second, the wave equation in the plane and in space, and traveling waves for it. Superposition and linearity. Visual illusions: plaids. The simplest approx to water waves (pics of this).

Third, Maxwell's equations – must not get bogged down in details here – light and EM waves and the speed of light. (Cable equation and story of the trans-Atlantic cable?) Modulation of radio/TV signals.