Define

$$\begin{aligned}\alpha(\Delta z) &= \alpha_{f,z_0}(\Delta_z)\\ &= \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} - f'(z_0)\end{aligned}$$

so that

$$\lim_{\Delta z\to 0}\alpha(\Delta z) = 0$$

After horsing around a bit, we can express the derivative as

$$f(z_0+\Delta z) = f(z_0) + f'(z)\Delta z + \alpha(\Delta z)\cdot\Delta z$$

In thinking about the derivative, we often ignore the α term, as it vanishes when $\Delta z \to 0$.

As an application, we can prove the chain rule. Suppose we have $f\circ g$, g is differentiable at z_0, and f differentiable at $w_0 = g(z_0)$. Then

$$g(z_0+\Delta z) = g(z_0) + \Delta z g'(z_0) + \Delta z\beta(\Delta z)$$

where $\lim_{\Delta z\to 0}\beta(\Delta z) = 0$, and similarly

$$f(w_0+\Delta w) = f(w_0) + \Delta w f'(w_0) + \Delta w\alpha(\Delta w).$$

Then

$$\begin{aligned}f(g(z_0+\Delta z)) &= f(g(z_0)) + f'(g(z_0))\cdot(\Delta z g'(z_0)+\Delta z\beta(\Delta z))\\ &\quad + (\Delta z g'(z_0)+\Delta z\beta(\Delta z))\alpha(\Delta z g'(z_0)+\Delta z\beta(\Delta z))\\ \frac{f(g(z_0+\Delta z))-f(g(z_0))}{\Delta z} &= \frac{1}{\Delta z}\left(f'(g(z_0))\cdot(\Delta z g'(z_0)+\Delta z\beta(\Delta z))\right.\\ &\quad \left.+(\Delta z g'(z_0)+\Delta z\beta(\Delta z))\alpha(\Delta z g'(z_0)+\Delta z\beta(\Delta z))\right)\\ \lim_{\Delta z\to 0}\frac{f(g(z_0+\Delta z))-f(g(z_0))}{\Delta z} &= \lim_{\Delta z\to 0} f'(g(z_0))(g'(z_0)+\beta(\Delta z)) + (g'(z_0)+\beta(\Delta z))(\alpha(\Delta z g'(z_0)+\Delta z\beta(\Delta z)))\\ &= f'(g(z_0))g'(z_0)\end{aligned}$$

since as $\Delta z \to 0$, the α and β terms vanish. *Lecture 13*

INTERPRETATION OF THE DERIVATIVE So, if we zoom in incredibly (infinitely near) z_0, we have $f(z_0+\Delta z) \approx f(z_0) + f'(z_0)\Delta z$. Draw this for various Δz. Note that multiplying by a number $f'(z_0)$ is just like scaling and rotating....