Homework 8

Due: Wednesday, April 11

1. [F] 7.3.10.

2. Recall that on $(-1, 1)$, we have the series expansion
$$\frac{1}{1-x} = \sum_{n \geq 0} x^n. \tag{1}$$
Find a power series expansion for $f(x) = \frac{1}{(1-x)^2}$ on $(-1, 1)$ in two different ways:
   (a) Using term by term differentiation on (1).
   (b) By explicitly squaring the series on the right-hand side of (1).
   (HINT: *As a warmup for (b), calculate* $(\sum_{n=0}^{5} x^n)^2$*. If* $N \geq j$*, what is the coefficient of* x^j *in* $(\sum_{n=0}^{N} x^n)^2$*?*)

3. Let f be a differentiable function such that $f(x+y) = f(x)f(y)$ for all real x and y. Show that
$$f'(x) = f'(0)f(x).$$
(HINT: *Calculate* $f'(x)$ *using the definition of the derivative.*)

4. In this problem, you'll need a tiny bit of information about complex numbers:
   - There is an absolute value function on $\mathbb{C}$ which extends the usual one, and still satisfies the triangle inequality; $|z + w| \leq |z| + |w|$.
   - $i^2 = -1$, and thus
   $$\begin{aligned} i^{4n+2} &= -1 \\ i^{4n+3} &= -i \\ i^{4n} &= 1 \\ i^{4n+1} &= i \end{aligned}$$
   for all integers n.

   Throughout this problem, suppose $\theta \in \mathbb{R}$.

   (a) Show that
   $$e(z) := \sum_{n \geq 0} \frac{z^n}{n!}$$
   defines a function on all of $\mathbb{C}$. (HINT: *It suffices to show that for any* $z \in \mathbb{C}$*, the (real, positive) series* $\sum |z|^n / n!$ *converges.*)
   (b) Show that
   $$\sum_{n \geq 0 \text{ even}} \frac{(i\theta)^n}{n!} = \cos(\theta).$$
   (HINT: *Look up, or derive, the Taylor expansion of* $\cos(\theta)$*.*)

(c) Show that
$$\sum_{n\geq 0 \text{ odd}} \frac{(i\theta)^n}{n!} = i\sin(\theta).$$

(d) Show that
$$e(i\theta) = \cos(\theta) + i\sin(\theta).$$

Go read David Mumford's account of a bank with imaginary interest (page 21).

5. Prove the trigonometric identities

$$\begin{aligned}\cos(a+b) &= \cos(a)\cos(b) - \sin(a)\sin(b)\\ \sin(a+b) &= \cos(a)\sin(b) + \sin(a)\cos(b)\end{aligned}$$

in two different ways:

(a) Using the characterization $\exp(ix) = \cos(x) + i\sin(x)$; and

(b) Using the uniqueness (from class, Thursday April 5) of solutions to the initial value problem

$$\begin{aligned}f'(x) &= -g(x)\\ g'(x) &= f(x)\\ f(0) &= a\\ g(0) &= b\end{aligned}$$