Homework 5

Due: Wednesday, February 22

1. Carefully explain how the general Stokes theorem:

   If $S \subset R \subset \mathbb{R}^n$ is a smooth k-dimensional manifold, and if $\omega \in \mathcal{A}^{k-1}(R)$ is a smooth $k-1$-form, then:
   $$\int_{\partial S} \omega = \int_S d\omega$$

   implies each of:

   (a) the fundamental theorem of calculus; (HINT: *How should one assign an orientation to the boundary of an oriented curve?*)

   (b) Green's theorem (Thm. 5.12 in [F]);

   (c) Gauss's theorem (Thm. 5.34 in [F]);

   (d) Stokes's theorem in $\mathbb{R}^3$ (Thm. 5.52 in [F]).

2. Suppose $S \subset R \subset \mathbb{R}^n$ is a k-manifold.

   (a) Show that if $\omega \in \mathcal{A}^{k-2}(R)$, then $\int_{\partial(\partial S)} \omega = 0$.

   (b) What do you think this says about the boundary of the boundary of a manifold? Explain.

3. Show that if $f \in \mathcal{A}^0(R)$, then $d \circ d(f) = 0$.

4. The diagram

   $$\mathcal{A}^0(R) \xrightarrow{d} \mathcal{A}^1(R) \xrightarrow{d} \mathcal{A}^2(R) \qquad (*)$$

   is called exact if, for every $\omega \in \mathcal{A}_1(R)$ such that $d\omega = 0$, there exists $f \in \mathcal{A}_0(R)$ such that $df = \omega$.

   (a) Suppose that R is simply connected. Why is (*) exact?

   (b) Let $R = \{(x, y) : \frac{1}{2} < x^2 + y^2 < 2\} \subset \mathbb{R}^2$. Show that (*) is *not* exact.