Lemma Summation by parts: Given a_n and b_n, set

$$a'_n = a_n - a_{n-1}$$
$$B_N = \sum_{0 \le n \le N} b_n.$$

Then

$$\sum_{n \le N} a_n b_n = a_N B_N - \sum_{1 \le n \le N} a'_n B_{n-1}$$

Proof Use the fact that $b_N = -B_{N-1} + B_N$. Using induction, we have

$$\begin{aligned}\sum_{n \le N} a_n b_n &= a_N b_N + \sum_{n \le N-1} a_n b_n \\ &= a_N(-B_{N-1} + B_N) + a_{N-1}B_{N-1} - \sum_{1 \le n \le N-1} a'_n B_{n-1} \\ &= -a'_N B_{N-1} + a_N B_N - \sum_{1 \le n \le N-1} a'_n B_{n-1} \\ &= a_N B_N - \sum_{1 \le n \le N} a'_n B_{n-1}\end{aligned}$$

as desired. □

Dirichlet's Theorem Suppose $\{a_n\}$, $\{b_n\}$ sequences with

1. a_n decreasing to zero, i.e., $a_{n+1} \le a_n$ and $\lim_{n\to\infty} a_n = 0$; and
2. The partial sums B_N are bounded in absolute value by some constant C, independent of N.

Then $\sum a_n b_n$ converges.

Example $\sum (-1)^n/n$; here, $a_n = \frac{1}{n}$, $b_n = (-1)^n$. Then $|B_N| \le 1$.

Proof We know the partial sums are given by

$$\sum_{n \le N} a_n B_n = a_N B_N - \sum_{1 \le n \le N} a'_n B_{n-1}.$$

Two claims: that $\lim_{N\to\infty} a_N B_N$ exists, and that $\sum_{1 \le n \le N} a'_n B_{n-1}$ converges.

For the first, $\lim_{N\to\infty} |a_N B_N| \le C \lim_{N\to\infty} |a_N| = 0$.