Does this work? $\alpha + \beta$

Last time: usual tests for convergence of series. (Root test, ratio test, integral test, limit comparison test.)

Integral test good for all $a_n \geq 0$, monotone decreasing (nonincreasing). To get it to work, need to interpolate the a_n with a function f: $f(n) = a_n$; $f(n)$ decreasing; f integrable.

Limit comparison: Given sequences $\{a_n\}$ and $\{b_n\}$, if $\lim_{n\to\infty} a_n/b_n$ exists and is (finite and) nonzero, then $\sum a_n$ converges if and only if $\sum b_n$ converges.

(Counter)example: If $a_n = (-1)^n/n$, and if $b_n = 1/n$, then $\sum a_n$ converges, but $\sum b_n$ diverges

Example: Suppose $b_n = g(n)$, where g is a rational function $g(x) = \frac{f(x)}{h(x)}$. For $n \gg 0$, $g(n)$ is always positive or always negative. Let $d = \deg(f) - \deg(h)$, and let $a_n = n^d$. Then

$$\lim_{n\to\infty} a_n/b_n$$

is nonzero, finite, exists. (Example: $f(x) = x^2 - 32$, $h(x) = -3x^3 + 17x + 9$; then $d = -1$;

$$a_n/b_n = \frac{1}{n} \cdot \frac{-3n^3 + 17n + 9}{n^2 - 32}$$

Note that $\sum b_n$ converges if and only if $-\sum b_n$ converges. The point is that $\lim_{n\to\infty} a_n/b_n = -3$; so $\sum b_n$ converges $\iff$ $\sum a_n$ converges; but $\sum \frac{1}{n}$ diverges.)

Absolute vs. conditional convergence

The series $\sum a_n$ is called absolutely convergent if $\sum |a_n|$ is convergent.

Lemma Absolute convergence implies convergence.

Proof Use the Cauchy criterion. Let s_N be the N^{th} partial sum for $\sum a_n$, i.e., $s_N = \sum_{n \leq N} a_n$, and let $s_N^* = \sum_{n \leq N} |a_n|$. Triangle inequality: if $j < k$, then

$$\begin{aligned} |s_k - s_j| &= \left| \sum_{j<n\leq k} a_n \right| \\ &\leq \sum_{j<n\leq k} |a_n| \end{aligned}$$

So, $|s_k - s_j| \leq \left| s_k^* - s_j^* \right|$. Since $\sum |a_n|$ converges, $\{s_n^*\}$ is Cauchy. Given $\epsilon > 0$, there exists N such that for $j, k > N$, $\left| s_j^* - s_k^* \right| < \epsilon$. So for such j and k, $|s_j - s_k| \leq \left| s_j^* - s_k^* \right| < \epsilon$, and $\{s_N\}$ is Cauchy. □

A series which converges, but not absolutely is conditionally convergent.

Example Define a_n by

$$a_n = \begin{cases} \frac{1}{n+1} & n \text{ even} \\ -\frac{1}{n} & n \text{ odd} \end{cases}$$

Then $\sum a_n = 1 - 1 + \frac{1}{3} - \frac{1}{3} + \frac{1}{5} - \frac{1}{5} \cdots$. The partial sums are given by

$$s_N = \begin{cases} 0 & N \text{ odd} \\ \frac{1}{N+1} & N \text{ even} \end{cases}$$

So $\lim_{N\to\infty} s_N = 0$, and the series converges. But $\sum |a_n| = 2 \sum \frac{1}{2n+1}$, which diverges. (Basic fact: $\sum \frac{1}{n}$ diverges)

Example $\sum \frac{(-1)^n}{n}$ converges conditionally
The convergence is a consequence of:

Theorem Suppose $\{a_n\}$ decreasing, positive, $\lim_{n\to\infty} a_n = 0$. Then

$$\sum (-1)^n a_n$$

converges.

Proof Consider the (odd) partial sums

$$\begin{aligned} s_{2N+1} &= a_0 + (-a_1 + a_2) + (-a_3 + a_4) + \cdots + (-a_{2N-1} + a_{2N}) - a_{2N+1} \\ &= a_0 + (\leq 0) + (\leq 0) + \cdots (\leq 0) - a_{2N+1} \\ &\leq a_0 \end{aligned}$$

But,

$$\begin{aligned} s_{2N+3} &= s_{2N+1} + (a_{2N+2} - a_{2N+3}) \\ &= s_{2N+1} + (\geq 0) \\ &\geq s_{2N+1} \end{aligned}$$

So, the sequence of odd partial sums $\{s_{2N+1}\}$ is increasing, and bounded, thus has a limit, L. Moreover, $s_{2N} = s_{2N+1} - a_{2N+1}$, and $\lim_{N\to\infty} a_{2N+1} = 0$, $\lim_{N\to\infty} s_{2N} = \lim s_{2N+1} = L$. □

Given a series $\sum a_n$, define

$$a_n^+ = \begin{cases} a_n & a_n > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$a_n^- = \begin{cases} -a_n & a_n < 0 \\ 0 & \text{otherwise} \end{cases}$$

Let $s_N^{\pm} = \sum_{n \le N} a_n^{\pm}$, and as above define $s_N^* = \sum_{n \le N} |a_n|$.

Then

$$\begin{aligned} s_N &= s_N^+ - s_N^- \\ s_N^* &= s_N^+ + s_N^- \end{aligned}$$

Theorem If $\sum a_n$ is absolutely convergent, then $\sum a_n^+$ and $\sum a_n^-$ are each convergent. If $\sum a_n$ conditionally convergent, then each of $\sum a_n^{\pm}$ is divergent.

Proof If $\sum a_n$ absolutely convergent, then $\lim s_N^*$ exists. Since $\sum |a_n| \ge s_N^* \ge s_N^{\pm}$, $\{s_N^+\}$ and $\{s_N^-\}$ are each increasing, bounded sequences, thus convergent.

For the converse, suppose $\sum |a_n|$ diverges. Then at least one of $\sum a_n^+$, $\sum a_n^-$ diverges. If $\sum a_n$ converges, and one of $\sum a_n^{\pm}$ converges, then the other must, too. So, if $\sum a_n$ conditionally convergent, then both $\sum a_n^{\pm}$ diverge.

Finite sums independent of order. What about infinite sums?

Let $\sigma : \mathbb{N} \to \mathbb{N}$ be a bijection (one-to-one, onto map; permutation). Given a sequence $\{a_n\}$, and such a σ, can define a rearrangement $\{b_n = a_{\sigma(n)}\}$. Want to compare convergence of $\sum a_n$ to that of $\sum b_n$

Target: If all $a_n \ge 0$, and $\{b_n\}$ a rearrangement of $\{a_n\}$, then $\sum a_n = \sum b_n$

If $A \subset \mathbb{N}$ is any finite subset, define a partial sum:

$$s_A = \sum_{n \in A} a_n.$$

Lemma If each $a_n \ge 0$, then

$$\sum a_n = \sup_{A \subset \mathbb{N} \text{ finite}} s_A$$

Notation: $I_N = \{1, 2, \cdots, N\}$

Proof We have

$$\begin{aligned} \sum a_n &= \lim_{N \to \infty} \sum_{1 \le n \le N} a_n \\ &= \lim_{N \to \infty} s_{I_N} \end{aligned}$$

$$\begin{aligned}\sum a_n &= \lim_{N\to\infty} \sum_{1\le n\le N} a_n \\ &= \lim_{N\to\infty} s_{I_N} \\ &= \sup_N s_{I_N} \\ &\le \sup_{A\subset\mathbb{N} \text{ finite}} s_A\end{aligned}$$

But each such $A \subset \mathbb{N}$ is contained in some I_N; and since all a_n positive, $s_A \le s_{I_N}$. So in $\sup_{A\subset\mathbb{N}}$, it suffices just to consider those sets of the form I_N. □