Given a symbol $\sum_{n\geq 0} a_n$, calculate partial sums $s_N = \sum_{n\leq N} a_n$, and say that $\sum_{n\geq 0} a_n = L$ if and only if $\lim_{N\to\infty} s_N = L$.

Most important series is the geometric one. Consider $\sum_n \alpha^n$. This has a limit if $|\alpha| < 1$. Key issue: $\alpha \neq 1$, then $s_N = \sum_{0\leq n\leq N} \alpha^n$ is $s_N = \frac{\alpha^N - 1}{\alpha - 1}$.

Series with positive terms.

Suppose all $a_i \geq 0$. Then either $\sum a_n = L$ for some finite L, or $\sum a_n = \infty$.

Comparison test: Suppose $0 \leq a_n \leq b_n$ for all n. Then

a. If $\sum b_n$ converges, then so does $\sum a_n$.

b. If $\sum a_n$ diverges, then so does $\sum b_n$.

Proof Let $s_N = \sum_{0\leq n\leq N} a_n$, $t_N = \sum_{0\leq n\leq N} b_n$. Then $\{s_N\}$ and $\{t_N\}$ are monotone increasing sequences. If $\sum b_N$ converges, then the t_N are bounded above (by $\sum b_N$).

Then $\{s_N\}$ is a bounded, nondecreasing sequence of numbers, thus $\{s_N\}$ has a limit, and $\sum a_n$ exists.

Reverse direction left as exercise. (If s_N unbounded, since $t_N \geq s_N$, t_N unbounded.) □

Integral tests:

Suppose $f : [1, \infty) \to \mathbb{R}$ positive, decreasing (nonincreasing) function, $f|_{[1,N)}$ integrable.

Theorem $\sum_{n\geq 1} f(n)$ converges if and only if $\int_1^\infty f(t)dt$ is finite.

(Recall that $\int_1^\infty f(t)dt$ means $\lim_{N\to\infty} \int_1^N f(t)dt$.)

Sketch. For $x \in [n, n+1]$,

$$f(n) \geq f(x) \geq f(n+1)$$
$$f(n) \geq \int_n^{n+1} f(x)dx \geq f(n+1)$$

Then

$$\sum_{1\leq n\leq N} f(n) \geq \int_1^{N+1} f(x)dx \geq \sum_{2\leq n\leq N+1} f(n).$$

(Key: $\int_1^{N+1} f(x)dx = \sum_{1\leq n\leq N} \int_n^{n+1} f(x)dx$.) Now take $\lim_{N\to\infty}$ of everything. If $\sum_{n\geq 1} f(n)$ diverges, then so does $\sum_{2\leq n} f(n)$, and then $\int_1^{N+1} f(x)dx$ gets arbitrarily large as $N \to \infty$.

Corollary Suppose $p \in \mathbb{R}$. Then $\sum_{n\geq 1} n^p$ converges $\iff$ $p < -1$.

Proof Consider $f(x) = x^p$ and its antiderivative:

$$F(x) = \begin{cases} \frac{x^{p+1}}{p+1} & p \neq -1 \\ \ln(x) & p = -1 \end{cases}$$

(Of course, $F(x) + c$ is also an antiderivative for f, for any constant c; but $\int_a^b F(x)dx = \int_a^b (F(x) + c)dx$)

Then

$$\int_1^N f(x)dx = \begin{cases} \frac{N^{p+1}}{p+1} - \frac{1}{p+1} & p \neq -1 \\ \ln(N) & p = -1 \end{cases}$$

From this, we see that $\lim_{N\to\infty} \int_1^N f(x)dx$ is finite if and only if $p < -1$.

Limit comparison test: Suppose $\{a_n\}$ and $\{b_n\}$ are positive sequences. Suppose $\lim_{n\to\infty} a_n/b_n = L$ exists and is positive (and finite). Then $\sum a_n$ converges if and only if $\sum b_n$ converges.

Proof There exists N such that for $n \geq N$, $L/2 < a_n/b_n < 2L$. So consider the sums $\sum_{n\geq N} a_n$ and $\sum_{n\geq N} b_n$. So, $\sum b_n$ converges if and only if $\sum_{n\geq N} b_n$ converges; but in the latter series, each $a_n < 2Lb_n$, so $\sum_{n\geq N} a_n < 2L \sum_{n\geq N} b_n$, and $\sum_{n\geq N} a_n$ converges.

Similarly, can show that if $\sum a_n$ converges, then so must $\sum b_n$.

Ratio Test If there exists r such that $a_{n+1}/a_n < r < 1$ for $n \gg 0$, then $\sum a_n$ converges.

If there exists R such that $a_{n+1}/a_n > R > 1$ for $n \gg 0$, then $\sum a_n$ diverges.

(If there exists $r < 1$ such that there exists N such that for $n \geq N$, $a_{n+1}/a_n < r$, then $\sum a_n$ converges.) ($n \gg 0$ means "for n sufficiently large")

Proof Compare to geometric series. □

Example Suppose that

$$a_n = \frac{n-1}{n^2 - 2n + 1}.$$

Then the ratio a_{n+1}/a_n is

$$a_{n+1}/a_n = \frac{n}{(n+1)^2 - 2(n+1) + 1} \frac{n^2 - 2n + 1}{n-1}$$

$$= \frac{n}{(n+1)^2} \frac{(n-1)^2}{n}$$

which is $\frac{(n-1)^2}{(n+1)^2}$. Unfortunately, $\lim_{n\to\infty}$ (that) is 1. So our tests can't decide the convergence of this series.

If instead, we took

$$b_n = \frac{n-1}{n^3-2n+1},$$

then the limit of the ratio of successive terms is