Homework 9

Due: Friday, April 9

1. Let $\{a_1, a_2, \cdots, a_n, \cdots\}$ be an enumeration of the elements in $\mathbb{Q} \cap [0,1]$.
   Define functions $f_n : [0,1] \to \mathbb{R}$ and $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} \frac{1}{n} & x = a_n \\ 0 & x \notin \mathbb{Q} \end{cases}$$
$$f_n(x) = \begin{cases} \frac{1}{k} & x = a_k, k \leq n \\ 0 & \text{otherwise} \end{cases}.$$

   (a) Show that each f_n is Riemann integrable.
   (b) Show that $f_n \to f$ uniformly on $[0,1]$. (HINT: *What is* $\|f_n - f\|_\infty$*?*)
   (c) Show that f is not Riemann integrable.

   In fact, f is Lebesgue *integrable, and* $\int_{[0,1]} f(x)dx = 0$.

2. Prove Theorem 7.13: Let $\{f_n\}$ be a sequence of continuous functions on $[a,b]$ such that the series $\sum_n f_n$ converges pointwise on $[a,b]$.

   (a) If $\sum_n f_n$ converges uniformly on $[a,b]$, then

$$\int_a^b \left(\sum_n f_n(x) \right) dx = \sum_n \int_a^b f_n(x)\, dx.$$

   (b) If each f_n is C^1 on $[a,b]$, and if $\sum_n f_n'$ converges uniformly on $[a,b]$, then

$$\frac{d}{dx} \left(\sum_n f_n(x) \right) = \sum_n f_n'(x).$$

3. [F]7.2.1.

4. [F] 7.3.2.

5. [F] 7.3.9.