Homework 7

Due: Friday, March 26

1. The goal of this problem is to study the convergence of

$$\sum_{n\geq 1} \frac{\sin(n)}{n^p}. \tag{1}$$

In this problem, do not *use the bounds from class on* $\sum^N \sin(n\theta)$.

   (a) Using the identity

   $$2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta),$$

   show that

   $$\sum_{1\leq n\leq N} 2\sin(n)\sin(\frac{1}{2}) = \cos(1-\frac{1}{2}) - \cos(N+\frac{1}{2}).$$

   (b) Show that there is a constant C such that for all N,

   $$\left|\sum_{1\leq n\leq N} 2\sin(n)\sin(\frac{1}{2})\right| < C.$$

   (c) For which p does $\sum_{n\geq 1}\frac{\sin(n)}{n^p}$ converge?

2. *Continuation of* (1). The goal of this problem is to show that sometimes the convergence in (1) is conditional.

   Consider the series $\sum \alpha_n$ and $\sum \beta_n$ given by

   $$\alpha_n = \frac{|\sin(n)| - |\sin(n-1)|}{2n}$$
   $$\beta_n = \frac{|\sin(n)| + |\sin(n-1)|}{2n}.$$

   (a) Show that $\sum_{n\geq 1}\alpha_n$ converges. (HINT: *Dirichlet again!*)

   (b) Use the identity

   $$\sin(\alpha-\beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$$

   to show that for all x, $\sin(1) \leq |\sin(x)| + |\sin(x-1)|$. (HINT: *Use* $\beta = x-1$.)

   (c) Show that $\sum_{n\geq 1}\beta_n$ diverges.

   (d) Show that $\sum_{n\geq 1}\frac{|\sin(n)|}{n}$ diverges. (HINT: *What is* $\alpha_n + \beta_n$*?*)

3. In this problem, you'll need a tiny bit of information about complex numbers:

   - There is an absolute value function $\mathbb{C}$ which extends the usual one, and still satisfies the triangle inequality; $|z+w| \leq |z| + |w|$.

- $i^2 = -1$, and thus

$$\begin{aligned} i^{4n+2} &= -1 \\ i^{4n+3} &= -i \\ i^{4n} &= 1 \\ i^{4n+1} &= i \end{aligned}$$

  for all integers n.

Throughout this problem, suppose $\theta \in \mathbb{R}$.

(a) Show that

$$e(z) := \sum_{n \geq 0} \frac{z^n}{n!}$$

defines a function on all of $\mathbb{C}$. (HINT: *It suffices to show that for any $z \in \mathbb{C}$, the (real, positive) series $\sum |z|^n / n!$ converges.*)

(b) Show that

$$\sum_{n \geq 0 \text{ even}} \frac{(i\theta)^n}{n!} = \cos(\theta).$$

(HINT: *Look up, or derive, the Taylor expansion of* $\cos(\theta)$.)

(c) Show that

$$\sum_{n \geq 0 \text{ odd}} \frac{(i\theta)^n}{n!} = i \sin(\theta).$$

(d) Show that

$$e(i\theta) = \cos(\theta) + i \sin(\theta).$$

Go read David Mumford's account of a bank with imaginary interest (page 21).

4. Let $\{\vec{v}^{(n)}\}_{n \geq 1}$ be a sequence of vectors in $\mathbb{R}^m$, and denote the coordinates of $\vec{v}^{(n)}$ by $v_1^{(n)}, \cdots, v_m^{(n)}$. Show that $\sum_{n \geq 1} \vec{v}^{(n)}$ exists if and only if, for each i, $\sum_{n \geq 1} v_i^{(n)}$ exists.

5. ~~[F]7.1.1~~