Homework 4

Due: Friday, February 19

1. Let $R \subset \mathbb{R}^2$ be the region
$$R = \{(x,y) : \frac{1}{2} < x^2 + y^2 < 2\}.$$
Consider the vector field
$$\vec{F} = (\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}).$$
Show that:
   (a) $\frac{\partial}{\partial x} F_2 = \frac{\partial}{\partial y} F_1$ on R, but
   (b) $\vec{F}$ is not integrable.

2. (a) [F] 5.8.4a. (HINT: *Green's theorem.*)
   (b) [F] 5.8.4b. (HINT: *Such a curve is contained in a circle.*)

On Monday, we will define the spaces of k-forms $\mathcal{A}_k(R)$ on an open set R of some $\mathbb{R}^n$, and the (exterior) derivative $d : \mathcal{A}_k(R) \to \mathcal{A}_{k+1}(R)$. In particular, we will have:

$$\mathcal{A}_0(R) \xrightarrow{d} \mathcal{A}_1(R) \xrightarrow{d} \mathcal{A}_2(R) \qquad (*)$$

3. Show that if $f \in \mathcal{A}_0(R)$, then $d \circ d(f) = 0$.

4. The diagram (*) is called exact if, for every $\omega \in \mathcal{A}_1(R)$ such that $d\omega = 0$, there exists $f \in \mathcal{A}_0(R)$ such that $df = \omega$.
   (a) Suppose that R is simply connected. Show that (*) is exact.
   (b) Let $R = \{(x,y) : \frac{1}{2} < x^2 + y^2 < 2\} \subset \mathbb{R}^2$. Show that (*) is *not* exact. (HINT: *See problem 1.*)