Homework 12
Due: Friday, May 7

1. [F]~~8.6.4ab~~ 8.2.4ab.

2. [F]8.6.3.

3. [F]8.6.4. (In this problem, the odd and even extensions of ϕ are defined by:

$$\phi_n^+(x) = \begin{cases} \phi(x) & x \in [0, \ell] \\ \phi(-x) & x \in [\ell, 0) \end{cases}$$
$$\phi_n^-(x) = \begin{cases} \phi(x) & x \in [0, \ell] \\ -\phi(-x) & x \in [\ell, 0). \end{cases}$$

)

4. Consider the space of square-integrable functions[1] on an interval $[a, b]$, and the following two notions of the "size" of a (real-valued) function:

$$\|f(x)\|_\infty = \sup_{c \in [a,b]} |f(c)|$$
$$\|f(x)\|_2 = \sqrt{\langle f(x), f(x) \rangle}$$
$$= \sqrt{\int_a^b |f(x)|^2 \, dx}.$$

Let $\{f_N\}$ be a sequence of functions on $[a, b]$, and let f be a function on $[a, b]$. Suppose that $f_N \to f$ with respect to $\|\cdot\|_\infty$, i.e., that

$$\lim_{N \to \infty} \|f_N - f\|_\infty = 0.$$

Show that $f_N \to f$ with respect to $\|\cdot\|_2$, i.e., that

$$\lim_{N \to \infty} \|f_N - f\|_2 = 0.$$

5. In the proof of Bessel's inequality in class, we showed that for $f(x)$ 2π-periodic and piecewise continuous,

$$\|f(x) - f_N(x)\|_2^2 = \|f(x)\|_2^2 - \|f_N(x)\|_2^2,$$

where

$$f_N(x) = \sum_{-N \le n \le N} c_n(f) \exp(inx)$$
$$c_n(f) = \langle f(x), \exp(inx) \rangle.$$

[1] Just assume that both f and its square are bounded and integrable, e.g., f is piecewise continuous and bounded.

Suppose you know that $f_N \to f$ uniformly. Prove that Bessel's inequality is actually an equality, i.e., that

$$\sum_{-\infty \leq n \leq \infty} |c_n|^2 = \frac{1}{2\pi} \|f(x)\|_2^2 .$$