Since the first midterm, we have covered material corresponding, essentially, to the following sections of the textbook:

Chapter 2 All except section 2.5.
Chapter 3 Sections 3.1, 3.2 and 3.3.
Chapter 4 Sections 4.1, 4.2 and 4.3.
The best way to study is to go back over course notes and homework. Here are some additional problems to think about. They're different from the typical homework problem, in that they often ask you to synthesize material from several different parts of the course.

1. Consider the function $f:[0,1] \rightarrow \mathbb{R}$ defined by

$$
f(x)= \begin{cases}0 & x=\frac{1}{n} \text { for some positive integer } n \\ x & \text { otherwise }\end{cases}
$$

(a) Let $D$ be the set of discontinuities of $f$, i.e,

$$
D=\{x \in[0,1]: f \text { is not continuous at } a\} .
$$

Describe the set $D$.
(b) Is $D$
i. open?
ii. closed?
iii. bounded?
iv. compact?
(c) Is $f$ integrable on $[0,1]$ ? If so, what is $\int_{0}^{1} f(x) d x$ ?
2. (a) Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function. Show that the following are equivalent:

- There exists a constant $A$ such that, for all $x$ and $y$ in $\mathbb{R}$,

$$
|f(x)-f(y)| \leq A|x-y|
$$

- There exists a constant $B$ such that, for all $z \in \mathbb{R}$,

$$
\left|f^{\prime}(z)\right| \leq B
$$

(b) Given an example of a function $g: \mathbb{R} \rightarrow \mathbb{R}$ and a constant $C$ such that, for all $x$ and $y$ in $\mathbb{R}$,

$$
|g(x)-g(y)| \leq C|x-y| ;
$$

but $g$ is not differentiable on all of $\mathbb{R}$.
3. Let $f: \mathbb{R}^{n} \rightarrow \mathbb{R}$ be a function which is differentiable at $\vec{a} \in \mathbb{R}^{n}$. Suppose $\vec{u} \in \mathbb{R}^{n}$ is a unit vector, i.e., $\|\vec{u}\|=1$.
(a) Show that $\partial_{\vec{u}} f=0 \Longleftrightarrow \vec{u}$ is perpendicular to the gradient vector $\vec{\nabla} f(\vec{a})$.
(b) Let $H$ be the Hessian of $f$ at $\vec{a}$. Show that $H \vec{u} \cdot \vec{u}$ is the second directional derivative of $f$ in the direction $\vec{u}$.
4. This problem gives the key calculations necessary to show that Newton's method works.

Suppose $f(x)$ is a function which is $C^{2}$ on $[a, b]$. Suppose there are positive numbers $m$ and $M$ such that $\left|f^{\prime}(x)\right|>m$ on $[a, b]$ and $\left|f^{\prime \prime}(x)\right|<M$ on $[a, b]$.
Further suppose that $f(a) f(b)<0$.
(a) Show that there is exactly one number $r \in(a, b)$ such that $f(r)=0$.
(b) Suppose $x_{0} \in[a, b]$. Show that there is a number $c$ between $x_{0}$ and $r$ such that

$$
-f\left(x_{0}\right)=f^{\prime}\left(x_{0}\right)\left(r-x_{0}\right)+\frac{1}{2} f^{\prime \prime}(c)\left(r-x_{0}\right)^{2}
$$

(Hint: Use a Taylor expansion for $f$ centered at $x_{0}$ to calculate $f(r)$.)
(c) Let

$$
x_{1}=x_{0}-\frac{f\left(x_{0}\right)}{f^{\prime}\left(x_{0}\right)} .
$$

Show that

$$
x_{1}-r=\frac{1}{2} \frac{f^{\prime \prime}(c)}{f^{\prime}\left(x^{\prime}\right)}\left(r-x_{0}\right)^{2}
$$

and that

$$
\left|x_{1}-r\right|<\frac{M}{2 m}\left|x_{1}-r\right|^{2}
$$

5. Consider the function

$$
F(x, y)=\ln \left(x^{2}+y^{2}+1\right)-x .
$$

The point $(0,0)$ is a solution to the equation $F(x, y)=0$.
(a) Near $(0,0)$, is it possible to solve for $y$ as a function of $x$ ?
(b) Near $(0,0)$, is it possible to solve for $x$ as a function of $y$ ?
(c) Find an equation for the tangent line to the graph of $F(x, y)=0$ at $(0,0)$.
6. Suppose that $f: \mathbb{R}^{2} \rightarrow \mathbb{R}, g: \mathbb{R}^{2} \rightarrow \mathbb{R}$ and $h: \mathbb{R}^{2} \rightarrow \mathbb{R}$ are all $C^{1}$ functions. Express the following two limits in terms of partial derivatives of these functions:
(a)

$$
\lim _{t \rightarrow 0} \frac{f(g(1+t, 2), h(1+t, 2))-f(g(1,2), h(1,2))}{t} .
$$

(b)

$$
\lim _{t \rightarrow 0} \frac{f(g(1,2)+t, h(1,2))-f(g(1,2), h(1,2))}{t}
$$

