□

We now exploit this give an upper bound for the number of primes all together. We'll prove this for the special case $X = 2^m$. We have

$$\pi(2^m) - \pi(2^{m-1}) \leq \frac{2^m}{\log}(2)\log(2^{m-1}) + O(1)$$
$$\pi(2^{m-1}) - \pi(2^{m-2}) \leq \frac{2^{m-1}\log(2)}{\log(2^{m-2}} + O(1)\cdots$$

Therefore, if we add up the number of primes in all these intervals, we find

$$\pi(2^m) \leq \sum_{j=2}^{m} \frac{2^j}{j-1} + O(\log(2^m))$$

I claim that there is a constant C such that for all m,

$$\pi(2^m) \leq C\frac{2^m}{m} + O(m)$$

It suffices to show the following:

Claim Suppose $C > 4$ is a constant such that for some $m_0 \geq 3$, $\sum_{j=2}^{m_0} \frac{2^j}{j-1} \leq C\frac{2^{m_0}}{m_0}$. Then the same inequality holds for all $m \geq m_0$.

To prove this, we proceed by induction on m: we have

$$\begin{aligned}
\sum_{j=2}^{m+1} \frac{2^j}{j-1} &= \sum_{j=2}^{m} \frac{2^j}{j-1} + \frac{2^{m+1}}{m} \\
&\leq C\frac{2^m}{m} + \frac{2^{m+1}}{m} \\
&= (\frac{C}{2}+1)\frac{2^{m+1}}{m} \\
&= \frac{C+2}{2m}(m+1)\frac{2^{m+1}}{m+1} \\
&\leq (C+2)\frac{2}{3}\frac{2^{m+1}}{m+1} \text{ since } m \geq 3 \\
&\leq C\frac{2^{m+1}}{m+1} \text{ since } C > 4.
\end{aligned}$$

□