Homework 1

Due: Monday, March 28

In class, on homework, and on exams, all vectors will be column vectors unless otherwise specified.

1. Let $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \right\} \subset \mathbb{R}^3$, and let $V = \operatorname{span} \mathcal{B}$. Let $\mathcal{C} = \left\{ \vec{c}_1 = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}, \vec{c}_2 = \begin{pmatrix} 3 \\ 4 \\ 15 \end{pmatrix} \right\}$.
   (You may assume that $\mathcal{B}$ and $\mathcal{C}$ are both linearly independent sets.)
   (a) Show that $\operatorname{span}(\mathcal{C}) = V$. Explain.
   (b) Determine the transition matrices:
      - P, from $\mathcal{B}$ to $\mathcal{C}$; and
      - Q, from $\mathcal{C}$ to $\mathcal{B}$.
   (c) Determine $[3\vec{c}_1 + 5\vec{c}_2]_{\mathcal{B}}$.

2. In each case, determine whether the given function is a linear transformation. Fully justify your answer.

   (a)

$$\mathbb{R}^3 \xrightarrow{f} \mathbb{R}^2$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} 2x_1 + x_2 \\ 4x_3 \end{pmatrix}$$

   (b)

$$\mathbb{R}^2 \xrightarrow{g} \mathbb{R}^2$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 + x_2 \\ x_1 \cdot x_2 \end{pmatrix}$$

   (c)

$$\mathbb{R}^2 \xrightarrow{h} \mathbb{R}^3$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} 3x_1 + 4x_2 \\ x_1 + x_2 \\ x_1 - x_2 \end{pmatrix}$$

(d)

$$\mathcal{P}_2 \xrightarrow{\quad j \quad} \mathbb{R}^3$$
$$ax^2+bx+c \longmapsto \begin{pmatrix} a+b \\ b-c \\ 2c \end{pmatrix}$$

3. For each of the linear transformations in the previous problem, find a basis for the null space and range, and find the nullity and rank.

4. Let $g : \mathcal{P}_3 \to \mathcal{P}_3$ be a linear transformation such that

$$g(1) = x^2+2; \qquad g(x) = x-3; \qquad g(x^2) = x^2+x+1.$$

What is $g(2x^2-5x+1)$?

5. Let $T : \mathcal{P}_2 \to \mathcal{P}_3$ be defined by $T(p(x)) = xp(x)$.

   (a) Show that T is a linear transformation.
   (b) What is the kernel (nullspace) of T?
   (c) What is the image (range) of T?

Bonus question Let A by an m by n matrix and B be be an n by k matrix.

a. Show that columnspace$(AB) \subseteq$ columnspace(A).

b. Show that rowspace$(AB) \subseteq$ rowspace(B).

c. Conclude that rank$(AB) \leq$ rank(A) and that rank$(AB) \leq$ rank(B).