Goal: Given some points in the plane, find a polynomial passing through them.

So much for that.

On to: abstract vector spaces!

Even though most of our calculations have been with matrices and vectors, we've really only used basic properties $\mathbb{R}^n$:

- Can add vectors together;
- Can multiply a vector by a number;
- Can divide by scalars (numbers).

First, need to define a *field*.

A ring is a set equipped with two operations, $\cdot$ and $+$, which behave the way you expect them to. (e.g., distributive law, associative law, etc.)

Example : $\mathbb{Z}$, $\mathbb{R}$, $\mathrm{Mat}_2(\mathbb{R})$

A ring is a field if multiplication is commutative ($a \cdot b = b \cdot a$), and if we can divide by any nonzero element.

Example $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}/5$ are all fields.

In contrast,

$\mathbb{Z}$ isn't a field, since we can't divide (without leaving the world of the integers)

$\mathbb{R}_{\geq 0}$ is not a field, since we can't subtract without (sometimes) leaving the set of nonnegative reals.

Now, let $\mathbb{F}$ be a field. A vector space over $\mathbb{F}$ is a set V such that you can add together elements of V to get a third element of V; and multiply an element of V by a number (in $\mathbb{F}$) to get an element of V.

Somewhat more precisely, we have maps

$$\mathbb{F} \times V \xrightarrow{\cdot} V$$

$$V \times V \xrightarrow{+} V$$

Properties:

- $(V, +)$ is an abelian group

  - If $u, v \in V$, then $u + v \in V$ (closure);
  - addition is associative $u + (v + w) = (u + v) + w$
  - V has an identity element $0 = 0_V$; $u + 0 = u$
  - additive inverses: if $v \in V$, there's a unique $-v \in V$ such that $v + (-v) = 0$;
  - commutative: $u + v = v + u$

- Multiplication distributes over addition: If $a \in \mathbb{F}$, $u, v \in V$, then $a \cdot (u + v) = a \cdot u + a \cdot v$
- Multiplication is associative: If $a, b \in \mathbb{F}$, and $v \in VW$, then $a \cdot (b \cdot v) = (a \cdot b) \cdot v$

Example

- $\mathbb{R}^n$ is a vector space (over $\mathbb{R}$)
- $\mathbb{F}^n$ is a vector space over $\mathbb{F}$
- Fix $m, n \in \mathbb{N}$. $\mathrm{Mat}_{m,n}(\mathbb{R})$ is a vector space over $\mathbb{R}$

  Note: If $A, B \in \mathrm{Mat}_{m,n}(\mathbb{R})$, then we can't multiply them; but that's okay. For $\mathrm{Mat}_{m,n}(\mathbb{R})$ to be a vector space, need to be able to add matrices; multiply a matrix by a number; and have this satisfy the axioms.
- $\mathcal{C}[0, 1]$ the set of continuous functions $[0, 1] \to \mathbb{R}$ is a vector space over $\mathbb{R}$.
- $\mathcal{C}^\infty(-\infty, \infty)$, the space of infinitely differentiable functions on $\mathbb{R}$, is a vector space over $\mathbb{R}$.
- $\mathcal{P}(\mathbb{R})[z]$ the set of polynomials, with real coefficients, in the variable z, is a vector space over $\mathbb{R}$.
- Fix $d \in \mathbb{N}$ Let $\mathcal{P}_d(\mathbb{R})[z]$ be the set of all polynomials, in the variable z, real coefficients, and degree at most d. This, too, is a vector space over $\mathbb{R}$.

  If $f(x), g(x)$ have degree at most d, their product fg is not necessarily an element of $\mathcal{P}_d(\mathbb{R})[z]$ But that's okay, since we only need to be able to add elements of our (alleged) vector space.

Subspaces:

Suppose V is a vector space over a field $\mathbb{F}$. A subspace $W \subset V$ is a nonempty subset which is a vector space in its own right.

Example In $\mathbb{R}^3$, consider

$$W = \{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} : x, y \in \mathbb{R} \}$$

Given $u, v \in W$, then their sum is also in W. And if $u \in W$, and $\lambda \in \mathbb{R}$, then $\lambda u \in W$, since its third coordinate is zero.

Example $\mathcal{P}_3(\mathbb{R})[z] \subset \mathcal{P}(\mathbb{R})[z]$ is a subspace of $\mathcal{P}(\mathbb{R})[z]$.

Theorem Given $W \subset V$ a subset of a vector space, W is a subspace (i.e., a vector space in its own right) if:

i. $0 \in W$;

ii. If $u, v \in W$ then $u + v \in W$;

iii. If $u \in W$ and $a \in \mathbb{F}$, then $a \cdot u \in W$.

Example We'll show that

$$W = \{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} : x, y \in \mathbb{R} \}$$

is a subspace of $\mathbb{R}^3$

Three things to check:

- $0_{\mathbb{R}^3} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, it's an element of W (use $x = y = 0$)
- Suppose $u, v \in W$. This means that there are x and y such that

  $$u = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

  and

  $$v = \begin{pmatrix} s \\ t \\ 0 \end{pmatrix}$$

  for some $s, t \in \mathbb{R}$ Then the sum of u and v is

  $$\begin{aligned} u + v &= \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} + \begin{pmatrix} s \\ t \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} x + s \\ y + t \\ 0 + 0 \end{pmatrix} \\ &\in W \end{aligned}$$

  since we were able to write it as (blah, blah, zero).

- Suppose $u \in W$, $a \in \mathbb{R}$. Then $au = \begin{pmatrix} ax \\ ay \\ a0 \end{pmatrix} \in W$