If A, B, C and $CA = B$, and C invertible, then A invertible if and only if B invertible.

Theorem If $A \in \mathrm{Mat}(n)$, and B is RREF(A), then A is invertible if and only if $B = I$.

Theorem A matrix A is invertible if and only if $\det(A) \neq 0$.

Proof As before, find C, a product of elementary matrices, such that $CA = RREF(B)$. Then:

$$E_r E_{r-1} \cdots E_1 A = B$$
$$\det(B) = \det(E_r E_{r-1} \cdots E_1 A)$$

Recall: If E elementary matrix, C square, then $\det(EC) = \det(E)\det(C)$.

$$= \det(E_r)\det(E_{r-1}) \cdots \det(E_1)\det(A)$$

So, $\det(B) = \text{nonzero} \det(A)$. So $\det(A) \neq 0 \iff \det(B) \neq 0 \iff B = I$. □

Theorem $A, B \in \mathrm{Mat}(n)$. Then $\det(AB) = \det(A)\det(B)$.

Proof If A and B are invertible, then so is AB (homework). By contrapositive, if AB is *not* invertible, then (at least one of) A or B isn't invertible. If this happens, then $\det(AB) = 0$, and one of $\det(A)$ or $\det(B)$ is zero. So the result holds...

Otherwise, suppose A, B, AB invertible. Since A is invertible, there are elementary matrices $E_1, \cdots, E_r$ such that

$$E_r E_{r-1} \cdots E_1 A = I$$
$$A = E_1^{-1} E_2^{-1} \cdots E_r^{-1} I$$
$$= F_1 \cdots F_r$$

So,

$$\det(AB) = \det(F_1 \cdots F_r B)$$
$$= \det(F_1)\det(F_2) \cdots \det(F_r) B$$

but each F_i elementary, so

$$= \det(F_1 \cdots F_r)\det(B)$$
$$= \det(A)\det(B)$$

**

Cramer's Rule

Example Consider the system of equations

$$5x + 3y = 8$$
$$2x + 7y = -1$$

Claim: The unique solution is:

$$x = \frac{\begin{vmatrix} 8 & 3 \\ -1 & 7 \end{vmatrix}}{\begin{vmatrix} 5 & 3 \\ 2 & 7 \end{vmatrix}}$$

$$y = \frac{\begin{vmatrix} 5 & 8 \\ 2 & -1 \end{vmatrix}}{\begin{vmatrix} 5 & 3 \\ 2 & 7 \end{vmatrix}}$$

Check: $\det(A) = 5 \cdot 7 - 3 \cdot 2 = 29$, $x = (8 \cdot 7 - (-1) \cdot 3)/29$, $y = (5 \cdot (-1) - 2 \cdot (8))/29$

So $x = 59/29$, $y = -21/29$. (Check – this works!)

In general, we have

Cramer's Theorem Given $A \in \mathrm{Mat}(n)$, $b \in \mathbb{R}^n$. Let $A^{(i)}$ be the matrix obtained from A by replacing the i^{th} column with b. Then the unique solution to

$$Ax = b$$

is given by

$$x_i = \frac{\det(A^{(i)})}{\det(A)}.$$

(Assumes $\det(A)$ is not zero, so that $Ax = b$ has a unique solution.)

Special case: $n = 2$.

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

Then

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

$$A^{(1)} = \begin{pmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{pmatrix}.$$

Similarly,

$$A^{(2)} = \begin{pmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{pmatrix}.$$

Then $x_i = \left|A^{(i)}\right| / |A|$. So there's a unique solution

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \det(A^{(1)})/\det(A) \\ \det(A^{(2)})/\det(A) \end{pmatrix}$$

Idea of proof:

Let $I^{(j)}$ be the matrix obtained from I by replacing j^{th} column with the vector x (the column vector $(x_1, \cdots, x_n)$).

Consider $AI^{(j)}$. The i^{th} column of the product is the i^{th} column of A (if $i \neq j$); and it's equal to Ax if $i = j$.

So, our equation $Ax = b$ (a vector equation) is the same as the matrix equation

$$AI^{(j)} = A^{(j)}.$$

But A is invertible, so

$$\det(A)\det(I^{(j)}) = \det A^{(j)}$$
$$\det I^{(j)} = \frac{\det A^{(j)}}{\det(A)}$$

But if we compute $\det I^{(j)}$, by expanding on the j^{th} row, we get $\det I^{(j)} = x_j$.

Invertibility

Suppose $A \in \text{Mat}(n)$. Then the following are equivalent:

- A is invertible.
- $RREF(A) = I$
- $\det(A) \neq 0$
- For each $b \in \mathbb{R}^n$, there's a unique solution to $Ax = b$
- The column space of A is $\mathbb{R}^n$

Recall: let $a^{(i)}$ be the i^{th} column of A. The column space of A is $\text{span}(a^{(1)}, \cdots, a^{(n)})$, i.e., the set of all linear combinations of the columns of A

Next goal: Given a collection of points in the plane, find a polynomial through them.

Example Find a cubic polynomial through $\{(1,5),(2,7),(4,8),(9,5),(11,12)\}$.

So, any cubic polynomial looks like

$$f(x) = a + bx + cx^2 + dx^3.$$

So we need to find $\{a,b,c,d\}$ such that $f(1) = 5$, $f(2) = 7$, etc.

This becomes the system

$$a + b \cdot 1 + c \cdot 1^2 + d \cdot 1^3 = 5$$
$$a + b \cdot 2 + c \cdot 2^2 + d \cdot 2^3 = 7$$
$$\text{etc.}$$