Homework 9

Due: Friday, April 21

1. Let V be an inner product space, and let $U \subseteq V$ be a subspace. Suppose that $U = \text{span}(u_1, \cdots, u_m)$. Let $v \in V$. Prove that $v \in U^\perp$ if and only if, for each $j = 1, \cdots, m$, $\langle u_j, v \rangle = 0$.

   Let V be an inner product space, and let $\mathcal{B} = \{v_1, \cdots, v_n\}$ be a basis for V. The associated Gram matrix is the $n \times n$ matrix G, whose ij entry is $g_{ij} = \langle v_i, v_j \rangle$. In class, we saw that

$$\langle u, w \rangle = ([u]_\mathcal{B})^T G [v]_\mathcal{B}.$$

2. Let V be an inner product space with basis $\mathcal{B}$. Let $U \subseteq V$ be $\text{span}(u_1, \cdots, u_m)$. Let A be the matrix

$$A = ([u_1]_\mathcal{B} \cdots [u_m]_\mathcal{B}).$$

   Suppose $v \in V$. Prove that $v \in U^\perp$ if and only if

$$A^T G [v]_\mathcal{B} = 0.$$

   Remark: In the special case where the basis is orthonormal, G is the identity matrix, so that the condition becomes $A^T [v]_\mathcal{B} = 0$.

3. *This problem explains another way to compute the orthogonal projection of a vector on to a subspace.* Let V be an inner product space, and let $U \subseteq V$ be a subspace with basis $\{u_1, \cdots, u_m\}$. As in the prevoius problem, let A be the matrix

$$A = ([u_1]_\mathcal{B} \cdots [u_m]_\mathcal{B}).$$

   Given a vector $v \in V$, our goal is to write $P_U(v) = c_1 u_1 + c_2 u_2 + \cdots + c_m u_m$.

   Before starting, convince yourself that if $P_U(v) = \sum_{i=1}^m c_i u_i$, then

$$A \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix} = [P_U(v)]_\mathcal{B}.$$

   (a) Prove that

$$A^T G([P_U(v)]_\mathcal{B} - [v]_\mathcal{B}) = 0.$$

   (Hint: $P_U(v) - v \in U^\perp$.)

(b) Show that the column vector

$$c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix}$$

satisfies

$$A^T G A c = A^T G [v]_{\mathcal{B}}.$$

4. Let $V = \mathbb{R}^4$ with the standard inner product. Let

$$U = \mathrm{span}\left(\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -2 \\ -2 \end{pmatrix} \right).$$

Find the vector in U which is closest to

$$v = \begin{pmatrix} 9 \\ 12 \\ -6 \\ -3 \end{pmatrix}$$

(Hint: Use the prevoius problem. What is G? What is A? At some point, you'll want to compute $(A^T A)^{-1}$.)