Homework 7

Due: Friday, March 31

1. Let $T \in \mathcal{L}(\mathbb{R}^2)$ be the operator whose matrix, with respect to the standard basis $\mathcal{E}$, is

$$[T]_{\mathcal{E}} = \begin{pmatrix} -7 & 10 \\ -5 & 8 \end{pmatrix}.$$

   (a) Find all eigenvalues of T.

   (b) For each eigenvalue, find an associated nonzero eigenvector.

   (c) Use your vectors from part (b) to make a new basis, $\mathcal{B}$, for $\mathbb{R}^2$. What is $[T]_{\mathcal{B}}$?

   (d) Show directly, by matrix multiplication, that

$$[\mathrm{id}]_{\mathcal{B}\leftarrow\mathcal{E}}[T]_{\mathcal{E}}\,\mathrm{id}_{\mathcal{E}\leftarrow\mathcal{B}} = [T]_{\mathcal{B}}.$$

2. Each of the following matrices describes a linear transformation from $\mathbb{R}^5$ to itself. In each case, is the matrix diagonalizable? (Equivalently, is there a basis in which the given linear transformation has diagonal matrix?)

   While you must explain your reasoning, you need not actually diagonalize any of the matrices.

   (a)

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ -1 & -72 & -316 & 0 & 97 \\ 48 & 222 & 950 & 0 & -291 \\ 10 & 36 & 147 & 3 & -45 \\ 144 & 666 & 2844 & 0 & -871 \end{pmatrix}$$

$$\mathrm{charpoly}_A(X) = (X-2)^3(X-3)^2$$

   (b)

$$B = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 14 & -3 & -5 & 0 & 0 \\ -14 & 5 & 7 & 0 & 0 \\ -28 & 10 & 8 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

$$\mathrm{charpoly}_B(X) = (X-2)^3(X-3)^2$$

(c)

$$C = \begin{pmatrix} 341 & -90 & 0 & -48 & 30 \\ 1348 & -356 & 0 & -102 & 120 \\ -50 & 12 & 3 & -4 & 7 \\ -112 & 30 & 0 & 21 & -10 \\ 46 & -12 & 0 & -4 & 6 \end{pmatrix}$$

$$\text{charpoly}_C(X) = (X-1)(X-2)(X-3)(X-4)(X-5)$$

3. The Fibonacci numbers are defined by the recurrence relation

$$F_n = \begin{cases} 1 & n = 0 \text{ or } 1 \\ F_{n-1} + F_{n-2} & \text{if } n \geq 2 \end{cases}$$

So, the first few Fibonacci numbers are $1, 1, 2, 3, 5, 8, 13....$

(a) For $n \geq 1$, let $v_n \in \mathbb{R}^2$ be the column vector $v_n = \begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix}$. So, the first few vectors are

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad v_3 = 32, \quad v_4 = 53...$$

Let A be the matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Show that $Av_n = v_{n+1}$.

(b) What are the eigenvectors and eigenvalues for A?

(c) Find a diagonal matrix D and a matrix P such that $D = P^{-1}AP$.

(d) Find a formula for A^n.

(e) Use your formula, and the fact that $v_{n+1} = A^n v_1$, to find a formula for F_n.

4. Which of the following define inner products on $\mathbb{R}^3$? (In each, x and y are arbitrary elements of $\mathbb{R}^3$, with $x = (x_1, x_2, x_3)^T$ and $y = (y_1, y_2, y_3)^T$.)

(a) $\langle x, y \rangle = x_1 y_1 + x_3 y_3$

(b) $\langle x, y \rangle = x_1 y_1 - x_2 y_2 + x_3 y_3$

(c) $\langle x, y \rangle = 2x_1 y_1 + x_2 y_2 + 4x_3 y_3$

(d) $\langle x, y \rangle = x_1 y_1^2 + x_2^2 y_2^2 + x_3^2 y_3 2$.

5. Suppose that V is a complex vector space, and that $\langle \cdot, \cdot \rangle$ is an inner product on V.

(a) Prove, using only the definition of an inner product (Axler, page 100), then for $\lambda \in \mathbb{C}$ and $u, v \in V$,

$$\langle u, \lambda v \rangle = \overline{\lambda} \langle u, v \rangle.$$

(b) Suppose that for all u and v, $\langle \lambda u, v \rangle = \langle u, \lambda v \rangle$. Prove that $\lambda \in \mathbb{R}$.