Homework 6
Due: Friday, March 24

1. We work with $\mathbb{R}^2$ with the standard basis $\mathcal{E} = \{(1,0)^T, (0,1)^T\}$. Let $T \in \mathcal{L}(\mathbb{R}^2)$ be the operator with matrix
$$[T]_{\mathcal{E}\leftarrow\mathcal{E}} = [T]_{\mathcal{E}} = \begin{pmatrix} 27 & 50 \\ 15 & -28 \end{pmatrix}.$$
Let v be the vector given by $[v]_{\mathcal{E}} = \begin{pmatrix} 1 & 0 \end{pmatrix}$.
   (a) Compute $[T(v)]_{\mathcal{E}}$ and $[T^2(v)]_{\mathcal{E}}$.
   (b) Find numbers a_0, a_1 and a_2 such that
$$a_2 \cdot T^2(v) + a_1 T(v) + v = 0.$$
   (c) Find an eigenvalue of T, and an associated eigenvector. (Hint: Consider the polynomial $a_2 z^2 + a_1 z + a_0$.)

2. Define $T \in \mathcal{L}(\mathbb{F}^3)$ by $T(x,y,z) = (2y, 0, 5z)$. Determine all eigenvalues and eigenvectors of T.

3. Suppose that $S, T \in \mathcal{L}(V)$ with $ST = TS$, and suppose that $\lambda \in \mathbb{F}$. Prove that $\operatorname{null}(T - \lambda\,\mathrm{id})$ is invariant under S.

4. Suppose that $T \in \mathcal{L}(V)$.
   (a) Suppose that $x, y \in V$ are nonzero vectors, and that x, y and $x + y$ are each eigenvectors of T. Prove that x and y belong to the *same* eigenvalue.
   (In other words, if $T(x) = \lambda_1 x$, $T(y) = \lambda_2 y$, and $T(x+y) = \lambda_3(x+y)$, then $\lambda_1 = \lambda_2$.)
   (b) Suppose that every element of V is an eigenvector of T. Prove that T is a scalar multiple of the identity operator.

5. Suppose $T \in \mathcal{L}(V)$ and $\dim \operatorname{im}(T) = k$. Prove that T has at most $k+1$ distinct eigenvalues.