Homework 4

Due: Friday, March 3

1. Let $A \in \mathrm{Mat}(m, n, \mathbb{F})$ be a matrix. For each of the following statements, indicate whether the statement is true or false. (For once, you need not justify your answer!)
   (a) If $\mathrm{rank}(A) = m$ then the system $Ax = b$ has at least one solution for every $b \in \mathbb{F}^m$.
   (b) If $\mathrm{rank}(A) = n$ then the system $Ax = b$ has at least one solution for every $b \in \mathbb{F}^m$.
   (c) If $\mathrm{rank}(A) = m$ then the system $Ax = b$ has at most one (i.e., no solution or a unique solution) for every $b \in \mathbb{F}^m$.
   (d) If $\mathrm{rank}(A) = n$ then the system $Ax = b$ has at most one (i.e., no solution or a unique solution) for every $b \in \mathbb{F}^m$.
   (e) If $Ax = b$, then $x \in \mathrm{row}(A)$.
   (f) If $Ax = b$, then $b \in \mathrm{col}(A)$.
   (g) If $B \in \mathrm{Mat}(m, n, \mathbb{F})$ is row equivalent to A then $\mathrm{col}(A) = \mathrm{col}(B)$.
   (h) If $B \in \mathrm{Mat}(m, n, \mathbb{F})$ is row equivalent to A then $\mathrm{row}(A) = \mathrm{row}(B)$.
   (i) If the columns of A are linearly independent then the rows of A are linearly independent.
   (j) If $\mathrm{col}(A) = \mathbb{F}^m$ and $m \leq n$ then $\mathrm{row}(A) = \mathbb{F}^n$.

2. Find two matrices A, B in $\mathrm{Mat}(2, 2, \mathbb{R})$ such that $A \cdot B \neq B \cdot A$.

3. Fix a matrix $B \in \mathrm{Mat}(n, n, \mathbb{F})$. Consider the function

$$\mathrm{Mat}(n, n, \mathbb{F}) \xrightarrow{f} \mathrm{Mat}(n, n, \mathbb{F})$$

$$A \longmapsto A \cdot B - B \cdot A$$

   Prove that this is a linear transformation, i.e., that $f \in \mathcal{L}(\mathrm{Mat}(n, n, \mathbb{F}), \mathrm{Mat}(n, n, \mathbb{F}))$.

4. Suppose that f and g are invertible elements of $\mathcal{L}(V, V) = \mathcal{L}(V)$ Prove that $f \circ g$ is invertible. What is the inverse of $f \circ g$?

5. Suppose $f \in \mathcal{L}(W, V)$ is a linear transformation between two finite-dimensional vector spaces. Let $\mathcal{C} = (w_1, \cdots, w_n)$ be a basis for W, and let $\mathcal{B}$ be a basis for V. Let $A = [f]_{\mathcal{B} \leftarrow \mathcal{C}}$. Show that the column space of A is equal to the image of f.

   (Concretely, suppose that $v \in V$. Show that $v \in \mathrm{im}(f)$ if and only if $[v]_{\mathcal{B}} \in \mathrm{col}(A)$.)