Homework 3

Due: Friday, February 24

1. Consider the following vectors in $\mathbb{R}^2$:

$$x = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad y = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad z = \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

In each case, is there a linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$ which maps the vectors x, y and z as prescribed? Explain. (Hint: $\{x, y\}$ is a basis for $\mathbb{R}^2$.)

   (a) $x \mapsto (2,3)^T$; $y \mapsto (0,1)^T$; $z \mapsto (1,5)^T$.

   (b) $x \mapsto (2,3)^T$; $y \mapsto (0,1)^T$; $z \mapsto (2,5)^T$.

   (c) $x \mapsto (1,1)^T$; $y \mapsto (0,1)^T$; $z \mapsto (3,3)^T$.

2. Suppose that $f \in \mathcal{L}(W, V)$ is surjective and that $w_1, \cdots, w_n$ spans W. Prove that $f(w_1), \cdots, f(w_n)$ spans V.

3. Prove that there does not exist a linear map from $\mathbb{R}^5$ to $\mathbb{R}^2$ whose null space is

$$U = \{(x_1, x_2, x_3, x_4, x_5)^T : x_1 = 3x_2, x_3 = x_4 = x_5\}.$$

(Hint: What is $\dim U$?)

4. Consider the following vectors in $\mathbb{R}^2$:

$$u_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad u_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \qquad v_1 = \begin{pmatrix} 14 \\ 22 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 9 \\ 14 \end{pmatrix}$$

Let $\mathcal{B} = (u_1, u_2)$ and $\mathcal{C} = (v_1, v_2)$; each one is a basis for $\mathbb{R}^2$.

   (a) Express u_1 and u_2 in terms of the basis $\mathcal{C}$, and do the same for v_1 and v_2 in terms of $\mathcal{B}$. That is, compute:

$$[u_1]_{\mathcal{C}}, [u_2]_{\mathcal{C}}, [v_1]_{\mathcal{B}}, [v_2]_{\mathcal{B}}.$$

   (b) What is $[\mathrm{id}]_{\mathcal{C} \leftarrow \mathcal{B}}$?

5. Recall that $\mathcal{P}_3 = \mathcal{P}_3(\mathbb{R})[z]$ is the vector space of polynomials of degree at most 3.

Let $\mathcal{B} = \{1, z, z^2, z^3\}$; let $\mathcal{C} = \{1, 2z, 3z^2, 4z^3\}$. Each is a basis of $\mathcal{P}_3$.

Let $D : \mathcal{P}_3 \to \mathcal{P}_3$ be the function $f(z) \mapsto f'(z)$.

   (a) Show that D is a linear transformation.

   (b) What is $[D]_{\mathcal{B} \leftarrow \mathcal{B}}$?

   (c) What is $[D]_{\mathcal{C} \leftarrow \mathcal{B}}$?