Homework 11

Due: Friday, May 5

1. Let V be an inner-product space.
   (a) Suppose that $T \in \mathcal{L}(V)$. Show that T^*T is self-adjoint.
   (b) Suppose that $S \in \mathcal{L}(V)$ satisfies $S^2 = T^*T$. Show that for all $v \in V$,
   $$\|Tv\| = \|Sv\|.$$
2. Let V be a finite-dimensional inner product space. Suppose that S and T are both isometries of V.
   (a) Prove that ST is an isometry of V.
   (b) Is it true that $S + T$ is an isometry of V? Prove or give a counterexample.
3. Suppose V is a finite-dimensional inner product space, $T \in \mathcal{L}(V)$ is self-adjoint, λ is a number, and $\epsilon > 0$. Suppose there exists a $v \in V$ such that $\|v\| = 1$ and
   $$\|Tv - \lambda v\| < \epsilon.$$
   Prove that T has an eigenvalue λ' such that $|\lambda - \lambda'| < \epsilon$. (HINT: *Start with an orthonormal basis of eigenvectors* $\{e_1, \cdots, e_n\}$ *for* V*, and express* v *and* T *in terms of those vectors. In fact, this result is also true for infinite-dimensional vector spaces.*)
4. Let $\mathcal{E}$ be the standard basis on $\mathbb{R}^3$. Let $T \in \mathcal{L}(\mathbb{R}^3)$ be the operator whose matrix, in the standard basis, is
   $$[T]_{\mathcal{E}} = \begin{pmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{pmatrix}$$
   Verify that the minimal polynomial of T is $(x-1)(x-2)$.
5. Let $\mathcal{E}$ be the standard basis on $\mathbb{R}^3$, and suppose that $T \in \mathcal{L}(\mathbb{R}^3)$ has
   $$[T]_{\mathcal{E}} = \begin{pmatrix} 0 & 0 & c \\ 1 & 0 & b \\ 0 & 1 & a \end{pmatrix}$$
   (a) What is the characteristic polynomial of T?
   (b) What is the minimal polynomial of T?