Homework 10

Due: Friday, April 28

1. Let V be an inner product space, and suppose that $S, T \in \mathcal{L}(V)$. Show that $(S+T)^* = S^* + T^*$.

2. Consider $\mathcal{P}_2(\mathbb{R})$, the vector space of real polynomials of degree at most 2. We will work with the usual basis $\mathcal{B} = \{1, x, x^2\}$ for $\mathcal{P}_2(\mathbb{R})$.

   Define an inner product by
   $$\langle p, q \rangle = \int_0^1 p(x)q(x)dx.$$
   Let T be the linear transformation
   $$T(a_2x^2 + a_1x + a_0) = a_1x.$$

   (a) Show that T is not self-adjoint. (By going back to the definition of the adjoint, you can do this without actually computing the full adjoint of T.)

   (b) Calculate the matrix $A = [T]_{\mathcal{B}}$.

   (c) The matrix you found is equal to its own conjugate transpose; $A = A^*$. Still, T is not self-adjoint. Explain why this it not a contradiction.

3. Suppose $T \in \mathcal{L}(V)$ is self-adjoint, and that λ and μ are distinct eigenvalues. Suppose that λ and μ are distinct eigenvalues of T, with corresponding eigenvectors u and v, respectively. Show that u and v are orthogonal.

4. Consider $V = \mathbb{R}^2$ with the standard inner product. Give an example of a linear transformation T and a one-dimensional subspace U such that U is invariant under T, but $U^\perp$ is *not* invariant under T.

5. *The following problem has nothing to do with adjoints.* Let V be a vector space, and $T \in \mathcal{L}(V)$.

   (a) Show that for all k, $\operatorname{null} T^k \supseteq \operatorname{null} T$.

   (b) Show that for all k, $\operatorname{im} T^k \subseteq \operatorname{im} T$.

   (Hint: Start with $k = 2$; the rest is easy.)