Homework 1

Due: Friday, February 3

Remark: Here is the notation we will use for certain fundamental vector spaces.

- $\mathcal{C}[0,1]$ is the set of continuous, $\mathbb{R}$-valued functions on the unit interval. $\mathcal{C}[a,b]$ is defined analogously.
- $\mathcal{C}^\infty[-\infty,\infty]$ is the set of infinitely differentiable functions from $\mathbb{R}$ to $\mathbb{R}$.
- $\mathcal{P}(\mathbb{R})[z]$ is the set of polynomials with real coefficients in a variable z.
- $\mathcal{P}_d(\mathbb{R})[z]$ is the set of polynomials with real coefficients in a variable z of degree at most d.

1. Let V be a vector space, and let U and W be subspaces.
   (a) Prove that the intersection $U \cap W$ is also a subspace of V.
   (b) Given an example showing that $U \cup W$ need not be a subspace of V.
2. Given an example of a vector space V and a subset U such that U is closed under scalar multiplication, but U is *not* a subspace of V.
3. Each of the following describes a subset S of a vector space V. (You may assume that V really is a vector space.) In each case, is S a subspace of V? Justify your answer.
   (a) $V = \mathbb{R}^3$, $S = \{(0,a,b) : a,b \in \mathbb{R}\}$.
   (b) $V = \mathbb{R}^3$, $S = \{(1,a,b) : a,b \in \mathbb{R}\}$.
   (c) $V = \mathcal{C}[0,5]$, $S = \{f \in V : f(2) = 3\}$.
   (d) $V = \mathcal{C}[0,5]$, $S = \{f \in V : f(2) = 0\}$.
   (e) $V = \mathcal{C}^\infty[-\infty,\infty]$, $S = \{f \in V : f'(x) - f''(x) = 0\}$. (HINT: *Don't compute the set S explicitly!*)
   (f) $V = \mathcal{C}[-1,1]$, $S = \{f \in V : f(-x) = f(x)\}$.
4. Prove that a finite list of vectors which contains the zero vector is linearly dependent.
5. Suppose that $v_1, \cdots, v_n$ are elements of a vector space. Prove that

$$\operatorname{span}(v_1 - v_2, v_2 - v_3, \cdots, v_{n-1} - v_n, v_n) = \operatorname{span}(v_1, v_2, \cdots, v_n).$$