A complete example of Gaussian elimination
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The point of Gaussian elimination is to make efficient use of row operations to transform a given (augmented) matrix into a row-equivalent matrix in reduced row-echelon form (RREF). (Recall that every matrix is row-equivalent to a unique matrix in RREF.)

Definition: A matrix is in **RREF** if:

1. All zero rows (rows with no nonzero entries) are at the bottom of the matrix;

2. The leftmost nonzero entry of each (nonzero) row is a 1 (called a *leading 1*);

3. The leftmost nonzero entry of each (nonzero) row is the *only* nonzero entry in its column; and

4. The matrix is in echelon form. See page 31 of the book for a technical definition of this. Simply put, it means that the leading 1's of the matrix progress from left to right as you move from the top of the matrix to the bottom. Algorithmically, put your pencil on the leading 1 of row 1. Move your pencil to the leading 1 of row 2. If it moved to the left, then the matrix is not in echelon form. If it moved to the right, proceed to the next row (and so on to the last nonzero row of the matrix). Note that a zero row cannot have a leading 1.

Recall that we are allowed to use (in any order) the following three row operations to get to the form we want:

1. Swap any two rows $(R_i \leftrightarrow R_j)$;

2. Multiply any row by a nonzero constant $\alpha \neq 0$ $(\alpha R_i \rightarrow R_i)$; and

3. Add to any row a nonzero multiple $\alpha \neq 0$ of another row $(R_j + \alpha R_i \rightarrow R_j)$.

So, how do we proceed? We could just guess what to do from one step to the next, but that may not be terribly efficient. Gaussian elimination is an algorithm (recipe) for doing this (other pseudocode for which may be found on pages 33-34 of the text):

<u>Algorithm</u>: **Gaussian elimination**
Input: A matrix A with m rows and n columns.
Output: A matrix that is both row-equivalent to A and in RREF.

For i from 1 to m (i is just the row number):

- Choose a row at or below row i which has the leftmost nonzero entry and (if it is not row i) swap that row with row i. If only zero rows (rows with all entries zero) remain, then stop – you are done.

- Make the leading nonzero entry of row i a 1 (by multiplying row i by the reciprocal of the leftmost nonzero entry).

- Use the leading 1 of row i to make 0 all other nonzero entries in the same (pivot) column as the leading 1.

- Swap any new zero rows down to the bottom of the matrix (maintaining echelon form).

Now for the example. Suppose we want to transform the following augmented matrix to reduced row-echelon form:

$$\left(\begin{array}{cccc|c} -\frac{1}{7} & -\frac{1}{7} & -1 & -\frac{12}{7} & -2 \\ 1 & 1 & 2 & 1 & 3 \\ -1 & -1 & -4 & -6 & -8 \\ -2 & 0 & -14 & -29 & -30 \end{array}\right)$$

Following the algorithm above, we start with $i = 1$ (i.e., we first consider the first row). Although the result is unique, the way in which we obtain it is not. We could multiply row 1 by -7 to get a leading 1, or we could just swap rows 1 and 2. I'll opt for the former option:

$$\left(\begin{array}{cccc|c} -\frac{1}{7} & -\frac{1}{7} & -1 & -\frac{12}{7} & -2 \\ 1 & 1 & 2 & 1 & 3 \\ -1 & -1 & -4 & -6 & -8 \\ -2 & 0 & -14 & -29 & -30 \end{array}\right) \quad \begin{array}{c} -7R_1 \to R_1 \\ \longrightarrow \end{array} \quad \left(\begin{array}{cccc|c} 1 & 1 & 7 & 12 & 14 \\ 1 & 1 & 2 & 1 & 3 \\ -1 & -1 & -4 & -6 & -8 \\ -2 & 0 & -14 & -29 & -30 \end{array}\right)$$

From here, we just use the new leading 1 (row 1, column 1) to make 0 everything else in column 1:

$$\left(\begin{array}{cccc|c} 1 & 1 & 7 & 12 & 14 \\ 1 & 1 & 2 & 1 & 3 \\ -1 & -1 & -4 & -6 & -8 \\ -2 & 0 & -14 & -29 & -30 \end{array}\right) \quad \begin{array}{c} R_2 - R_1 \to R_2 \\ \longrightarrow \end{array} \quad \left(\begin{array}{cccc|c} 1 & 1 & 7 & 12 & 14 \\ 0 & 0 & -5 & -11 & -11 \\ -1 & -1 & -4 & -6 & -8 \\ -2 & 0 & -14 & -29 & -30 \end{array}\right)$$

$$\begin{array}{c} R_3 + R_1 \to R_3 \\ \longrightarrow \end{array} \quad \left(\begin{array}{cccc|c} 1 & 1 & 7 & 12 & 14 \\ 0 & 0 & -5 & -11 & -11 \\ 0 & 0 & 3 & 6 & 6 \\ -2 & 0 & -14 & -29 & -30 \end{array}\right) \quad \begin{array}{c} R_4 + 2R_1 \to R_4 \\ \longrightarrow \end{array} \quad \left(\begin{array}{cccc|c} 1 & 1 & 7 & 12 & 14 \\ 0 & 0 & -5 & -11 & -11 \\ 0 & 0 & 3 & 6 & 6 \\ 0 & 2 & 0 & -5 & -2 \end{array}\right)$$

Things are looking good at this point, at least in column 1. We have our leading 1 in row 1 (column 1 is the pivot column corresponding to that leading 1), and the rest of the column has been zeroed out.

Now we increment i to 2. We want the leftmost nonzero entry of row 2 to be as far left as possible (using swapping). Of rows 2, 3, and 4, row 4 has the leftmost nonzero entry, so we swap that up to row 2:

$$\left(\begin{array}{cccc|c} 1 & 1 & 7 & 12 & 14 \\ 0 & 0 & -5 & -11 & -11 \\ 0 & 0 & 3 & 6 & 6 \\ 0 & 2 & 0 & -5 & -2 \end{array}\right) \quad \begin{array}{c} R_2 \leftrightarrow R_4 \\ \longrightarrow \end{array} \quad \left(\begin{array}{cccc|c} 1 & 1 & 7 & 12 & 14 \\ 0 & 2 & 0 & -5 & -2 \\ 0 & 0 & 3 & 6 & 6 \\ 0 & 0 & -5 & -11 & -11 \end{array}\right)$$

Multiplying by $\frac{1}{2}$ makes the leading entry of row 2 a 1. After that, we need to make zero only the 1 directly above our new leading 1. Notice that the 0 before the leading 1 in row 2 means that we will not damage the special structure of column 1 while making column 2

into the format that we want. (The fact that the two entries directly below our new leading 1 are both zero already is just dumb luck.)

$$\tfrac{1}{2}R_2 \to R_2 \atop \longrightarrow \left(\begin{array}{cccc|c} 1 & 1 & 7 & 12 & 14 \\ 0 & 1 & 0 & -\tfrac{5}{2} & -1 \\ 0 & 0 & 3 & 6 & 6 \\ 0 & 0 & -5 & -11 & -11 \end{array} \right) \quad {R_1 - R_2 \to R_1 \atop \longrightarrow} \left(\begin{array}{cccc|c} 1 & 0 & 7 & \tfrac{29}{2} & 15 \\ 0 & 1 & 0 & -\tfrac{5}{2} & -1 \\ 0 & 0 & 3 & 6 & 6 \\ 0 & 0 & -5 & -11 & -11 \end{array} \right)$$

The leading 1 of row 2 satisfies all of the properties we need, so we increment i again, to 3. For row 3, we can choose to keep it or swap for row 4 (never up – only down!). There is no point in doing that, so we just make the leading nonzero entry a 1 and eliminate all nonzero entries above and below this new leading 1. (Once again, we have a fortunate fluke, this time a zero in the $(3, 2)$ coordinate.)

$$\tfrac{1}{3}R_3 \to R_3 \atop \longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 7 & \tfrac{29}{2} & 15 \\ 0 & 1 & 0 & -\tfrac{5}{2} & -1 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & -5 & -11 & -11 \end{array} \right) \quad {R_1 - 7R_3 \to R_1 \atop \longrightarrow} \left(\begin{array}{cccc|c} 1 & 0 & 0 & \tfrac{1}{2} & 1 \\ 0 & 1 & 0 & -\tfrac{5}{2} & -1 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & -5 & -11 & -11 \end{array} \right)$$

$$R_4 + 5R_3 \to R_4 \atop \longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & \tfrac{1}{2} & 1 \\ 0 & 1 & 0 & -\tfrac{5}{2} & -1 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & -1 & -1 \end{array} \right)$$

Almost done. Before we tackle that last row, pretend that it is a zero row. In that case, we would be done, and there would be infinitely many solutions (since there are non-pivot columns). Now pretend that row 4, instead of a zero row, is a row with zeros to the left of the bar and a 1 to the right. In that case, we would still need to use that 1 (a leading 1) to zero out the remainder of column 5. However, the point is that such a row would correspond to the linear equation $0 = 1$. This has no solutions, so (in this case), the solution set of the linear system corresponding to the augmented matrix would have no solutions.

Enough pretending – back to our problem. Clearly, we increment i to 4, multiply row 4 by -1, and use our new leading 1 to zero out all other entries of column 4:

$$-R_4 \to R_4 \atop \longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & \tfrac{1}{2} & 1 \\ 0 & 1 & 0 & -\tfrac{5}{2} & -1 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right) \quad {R_1 - \tfrac{1}{2}R_4 \to R_1 \atop \longrightarrow} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \tfrac{1}{2} \\ 0 & 1 & 0 & -\tfrac{5}{2} & -1 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right)$$

$$R_2 + \tfrac{5}{2}R_4 \to R_2 \atop \longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \tfrac{1}{2} \\ 0 & 1 & 0 & 0 & \tfrac{3}{2} \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right) \quad {R_3 - 2R_4 \to R_3 \atop \longrightarrow} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \tfrac{1}{2} \\ 0 & 1 & 0 & 0 & \tfrac{3}{2} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right)$$

We are done! At this point, it would be wise to check that this is indeed reduced row-echelon form. The (unique) solution of the linear system corresponding to the original augmented matrix is then $\left(\tfrac{1}{2}, \tfrac{3}{2}, 0, 1 \right)$.