Homework 8

Due: *No.*

No homework officially due this week; but working through these problems, as well as going over earlier homework, might help you study for the midterm.

1. *Converse to HW7#6* Let R be a commutative ring with identity, and let $I \subseteq R$ be an ideal. Suppose R/I is not an integral domain. Show that there are elements $a, b \in R$ such that $a \notin I$, $b \notin I$, but $ab \in I$.

   An ideal P is called a prime ideal if, whenever $ab \in P$, at least one of a or b is in P. We have shown that R/P is an integral domain if and only if P is prime.

2. Let F be a field, let $g(x) \in F[x]$ be a polynomial of degree $d \geq 1$, and let $I = g(x)F[x]$.

   Show that if $f(x) \in F[x]$, then there is a unique polynomial $r(x)$ such that $\deg r < d$ and $[f(x)] = [r(x)]$ in $F[x]/I$.

3. Consider the two rings

$$R = \mathbb{Z}_3[x]/(x^2+1);$$
$$S = \mathbb{Z}_5[x]/(x^2+1).$$

   (a) How many elements does R have? How many elements does S have? (HINT: *Use* (2))

   (b) Exactly one of R and S is an integral domain. Which is it? Why? (HINT: *Use* (1).)
   In fact, we showed in class that any finite integral domain is actually a field!

4. Consider the ring homomorphism

$$\mathbb{Q}[x] \xrightarrow{\phi} \mathbb{R}$$
$$f(x) \longmapsto f(\sqrt{2}).$$

   (a) Is ϕ injective? Surjective?

   (b) What is $\ker\phi$? Prove your answer is correct.

   (c) Consider the map $\overline{\phi} : \mathbb{Q}[x]/(\ker\phi) \to \mathbb{R}$. Is $\overline{\phi}$ injective? Surjective?

5. [J]40(c).