Homework 1

Due: Monday, September 8

1. For a natural number n, draw n generic* lines. Let $R(n)$ be the resulting number of regions in the plane.
   (a) Calculate $R(n)$ for $n \in \{1, 2, 3\}$.
   (b) It turns out that R is represented by a quadratic polynomial; there are numbers a, b and c such that
   $$R(n) = an^2 + bn + c.$$
   Use the values of $R(1)$, $R(2)$ and $R(3)$ to find three linear relations satisfied by a, b and c.
   (c) Solve for a, b and c.
   (d) Compute $R(4)$ by hand, and verify that it equals
   $$a \cdot 4^2 + b \cdot 4 + c.$$

2. (a) Euler considered the polynomial
   $$E(x) = x^2 - x + 41.$$
   Compute $E(n)$ for some small integers n. What do you notice about the primality of $E(n)$?
   (b) For a fixed integer k, define the polynomial
   $$E_k(x) = x^2 - x + k.$$
   Prove that $E_k(k)$ is *never* prime.
   EXTRA: Do you think there is a nonconstant polynomial $p(x)$ such that $p(n)$ is *always* prime? Explain.

3. The Fibonacci numbers F_n are defined by
   $$F_0 = 0$$
   $$F_1 = 1$$
   $$F_n = F_{n-2} + F_{n-1} \text{ if } n \geq 2.$$
   Define
   $$T_n = F_n^2 + F_{n+1}^2.$$
   (a) Compute T_n for $n \in \{1, \cdots, 10\}$.
   (b) Conjecture a simple formula for T_n.
   Extra Can you prove your conjecture?

*The lines are distinct; no pair of lines is parallel; no three lines meet at a single point.

4. Let $\triangle$ be a right triangle with side lengths a and b, and hypotenuse length c. Prove the Pythagorean theorem, as follows.
   (a) Draw a square $\square$ whose sides have length $a+b$. Divide this $\square$ into four copies of $\triangle$ and a square of side length c.
   (b) Calculate the area of $\square$ in two different ways.
   (c) Use this this to show that
   $$a^2 + b^2 = c^2.$$
5. Consider a circle of diameter d; let A be its area.
   (a) By inscribing the circle in a square of side length d, find an upper bound for A.
   (b) By inscribing a square inside the circle, find a lower bound for A.
   (c) In problem 10 of the Moscow papyrus,† A is approximated by the area of a square whose sides have length $\frac{8}{9}$. How does this estimate compare to the estimates in parts (a) and (b)?

† An Egyptian manuscript, 3700 years old