MPE Review

Section II: Trigonometry

Review similar triangles, right triangles, and the definition of the sine, cosine and tangent functions of angles of a right triangle. In particular, recall that the corresponding sides of similar triangles are proportional. Moreover, two triangles are similar if two angles of one are equal to two angles of the other. Don't overlook the enormously important Pythagorean Theorem for right triangles, ($a^2 + b^2 = c^2$).

You will need to use a scientific calculator for the trigonometry section of the Colorado State University Mathematics Placement Examination. In particular, review how and when to use radian or degree measure for angles, and how to evaluate trigonometric and inverse trigonometric functions.

ANGLES

The angle θ shown in the Figure T1 is in **standard position**; *i.e.*, its initial side is the positive *x*-axis and its terminal side is a ray from the origin. The angle is **positive** if the rotation is counter-clockwise, and **negative** if the rotation is clockwise.

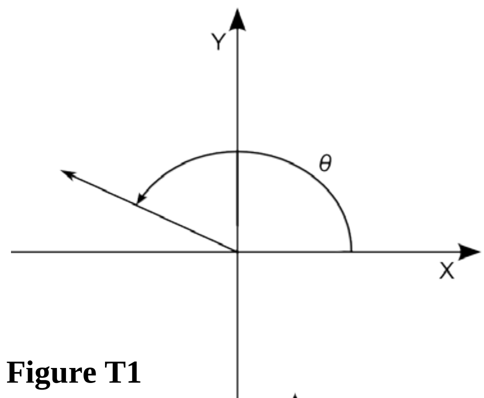


Figure T1

Example: Estimate the degree and radian measures of the angle θ in Figure T1.

Solution: Angle θ is formed by rotating a ray through $^5/_{12}$ revolutions counterclockwise from the positive *x*-axis. One full revolution counterclockwise forms an angle of 360° or 2π radians. Therefore, the measures of angle θ are

$$\frac{5}{12}(360°) = 150° \text{ or } \frac{5}{12}(2\pi) = \frac{5\pi}{6}.$$

Problems:

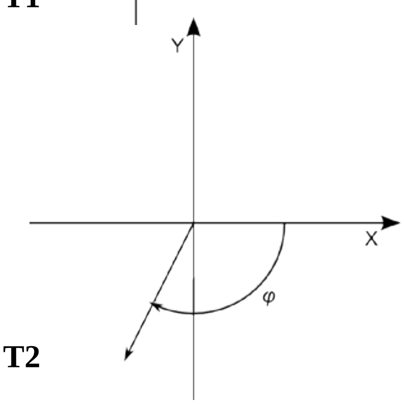


Figure T2

1. What is the approximate degree measure of the angle in Figure T2?

2. What is the approximate degree and radian measures of the angle in Figure T3?

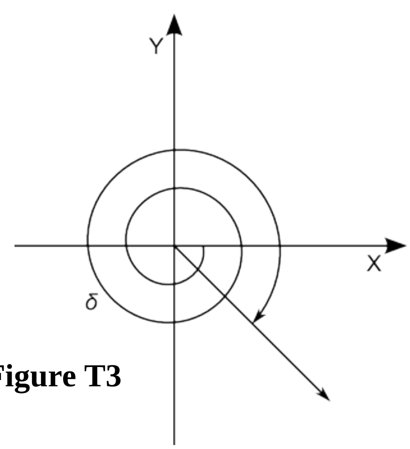


Figure T3

Since one counterclockwise revolution forms an angle of 360° or 2π radians, the equations for converting between degrees and radians are

$$1° = \frac{2\pi}{360} \text{ radians} = \frac{\pi}{180} \text{radians}$$

and

$$1 \text{ radian} = \frac{360}{2\pi} \text{ degrees} = \frac{180}{\pi} \text{degrees.}$$

Problems:

3. What is the radian measure of 108° angle?

4. What is the degree measure of an angle of $\frac{7\pi}{18}$ radians?

5. What is the degree measure of an angle of $-\frac{5\pi}{17}$ radians to the nearest tenth of a degree?

ARC LENGTH

If an arc of length s on a circle of radius r subtends a central angle of radian measure θ, then

$$s = r\theta.$$

Example: Consider a circle with radius 4.
What is the arc length intercepted by a 45° angle?

Solution: First convert to radians: $45° = \frac{\pi}{4}$.

And so with $\theta = \frac{\pi}{4}$ we have

$$s = r\theta$$

$$s = 4\left(\frac{\pi}{4}\right) = \pi.$$

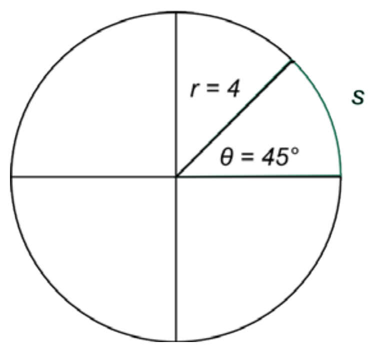


Figure T4

Problems:

6. Determine the arc length intercepted by a 78° angle in a circle of radius 3.5.

7. Determine the arc length intercepted by $\frac{9\pi}{5}$ radians in a circle of radius 4.

AREA OF A SECTOR

Consider the sector determined by the angle θ, in the circle with radius, r. The arc length of a 180° is π.
Recall that the area of the circle is given by $A = \pi r^2$.
Since the sector is the fractional part $\frac{\theta}{2\pi}$ of the circle,
the area of the sector then is given by

$$A = \frac{\theta}{2\pi} \cdot \pi r^2 = \frac{r^2\theta}{2}.$$

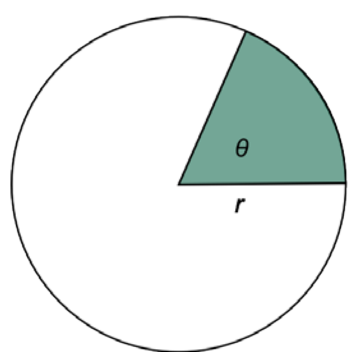


Figure T5

Example: Consider a circle with radius 4. What is the area of a sector determined by a 45° angle?

Solution: First convert to radians: $45° = \frac{\pi}{4}$.

And so with $\theta = \frac{\pi}{4}$ we have

$$A = \frac{(4)^2 \cdot \frac{\pi}{4}}{2} = 2\pi.$$

Problems:

8. Find the area of a sector determined by a 196° angle in a circle of radius 6.

9. Find the area of a sector determined by $\frac{5\pi}{3}$ radians in a circle of radius 2.

THE TRIGONOMETRIC FUCTIONS

For an angle θ in standard position, let (x, y) be a point on its terminal side. By the Pythagorean Theorem $r = \sqrt{x^2 + y^2}$.

The six trigonometric functions of θ are defined as

$$\sin\theta = \frac{y}{r}, \qquad \csc\theta = \frac{r}{y},$$

$$\cos\theta = \frac{x}{r}, \qquad \sec\theta = \frac{r}{x},$$

$$\tan\theta = \frac{y}{x} \quad \text{and} \quad \cot\theta = \frac{x}{y}.$$

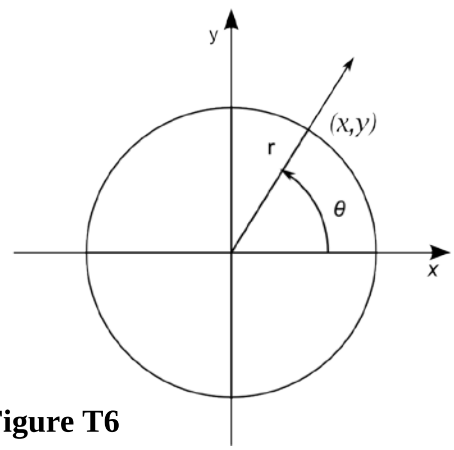


Figure T6

For an acute angle θ of a right triangle, as shown in Figure T7, the trigonometric functions can also be defined in terms of the side opposite angle θ, the side adjacent to angle θ, and the hypotenuse. Thus,

$$\sin\theta = \frac{\text{side opposite } \theta}{\text{hypotenuse}}, \qquad \csc\theta = \frac{\text{hypotenuse}}{\text{side opposite } \theta},$$

$$\cos\theta = \frac{\text{side adjacent to } \theta}{\text{hypotenuse}}, \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{side adjacent to } \theta},$$

$$\tan\theta = \frac{\text{side opposite } \theta}{\text{side adjacent to } \theta} \quad \text{and} \quad \cot\theta = \frac{\text{side adjacent to } \theta}{\text{side opposite } \theta}.$$

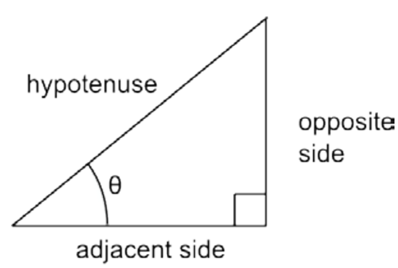


Figure T7

Notice that cot θ, sec θ, and csc θ are respectively the reciprocals of tan θ, cos θ and sin θ, and that the fundamental identity

$$\sin^2\theta + \cos^2\theta = 1$$

is just another way of expressing the Pythagorean Theorem.

By examining the various cases, we can see that the sine function is positive for angles whose terminal side is in the first or second quadrant and negative for angles in quadrants III and IV. Similarly, the cosine is positive in quadrants I and IV and negative in quadrants II and III; while the tangent is positive in quadrants I and III and negative in quadrants II and IV.

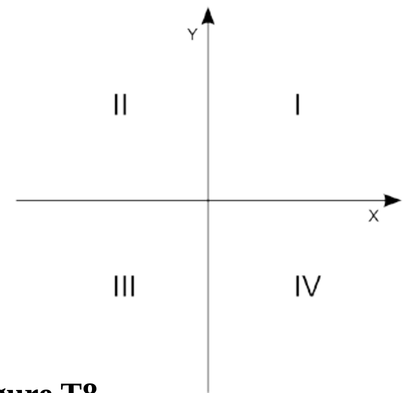


Figure T8

Example: The angle ϕ is a quadrant II angle and tan $\phi = -\frac{5}{8}$. What is the value of $\cos\phi$?

Solution: Draw a figure so that ϕ has its terminal side in quadrant II and so that $\tan\phi = \frac{5}{-8} = -\frac{5}{8}$.

Then ϕ

$$r = \sqrt{x^2 + y^2} = \sqrt{(5)^2 + (-8)^2} = \sqrt{89}.$$

Hence,

$$\cos\phi = \frac{-8}{\sqrt{89}} = \frac{-8\sqrt{89}}{89}.$$

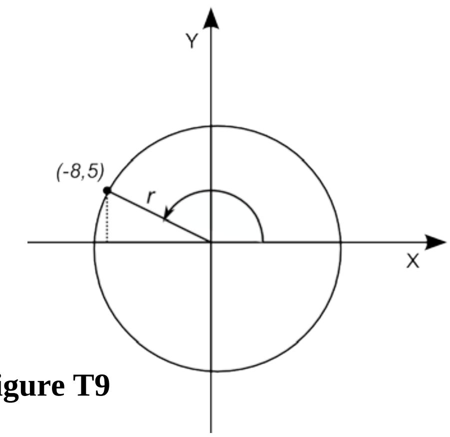


Figure T9

Example: Find all angles θ between 0 and 2π such that tan θ = -2.7933.

Solution: We need to determine the quadrants of the two angles θ_1 and θ_2 between 0 and 2π such that tan θ = -2.7933. Since tan θ = -2.7933 is negative and the tangent is negative in quadrants II and IV, θ_1 is in quadrant II and θ_2 is in quadrant IV.

Next we need to find the reference angle for θ_1 and θ_2.

$$\tan\theta' = |-2.7933| = 2.7933$$
$$\theta' = 1.2270$$

Make a sketch of the angles θ_1 and θ_2 in the correct quadrants with reference angle $\theta' = 1.2270$ as in Figure T10.

Determine θ_1 and θ_2 using Figure T10 as a guide:

$$\theta_1 = \pi - \theta' = \pi - 1.2270 = 1.9146$$

and

$$\theta_2 = 2\pi - \theta' = 2\pi - 1.2270 = 5.0562$$

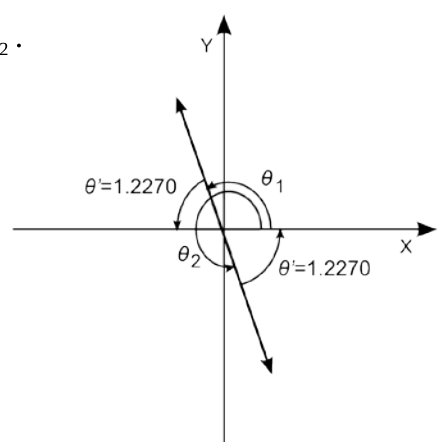


Figure T10

Problems:

10. Given $\sin\theta = \frac{-2}{9}$ and $\cot\theta$ is negative, what is the value of

a) $\cos\theta$ (exactly)?

b) $6\cos\theta - 2\cot\theta$ (rounded to two decimal places)?

In Problems 11-15, use a calculator to approximate the answer to five decimal places.

11. Given that angle ϕ is a quadrant II angle and $\tan\phi = -\frac{5}{8}$, find $6\sin\phi + 5\sec\phi$.

12. Angle θ is a quadrant III angle and $\csc\theta = -\frac{9}{5}$. What is the value of $\tan\theta - 5\sec\theta$?

13. The point (0,-5) is on the terminal side of angle θ in standard position.
What is the value of $2\tan\theta + 5\csc\theta$?

14. What is the value of $3\sin(864°)\sec(164°) + 8\tan(-196°)$?

15. What is the value of $2\sin\left(\frac{\pi}{11}\right)\csc(1.43) + 7\cos\left(-\frac{20\pi}{7}\right)$?

16. Angles α and β are between 0° and 360° with $\cos\alpha = -0.9063$ and $\tan\beta = -0.4877$.
Which of the following is a possible (approximate) value for $\alpha + \beta$?

A) 51° **B)** 129° **C)** 309° **D)** 361° **E)** 669°

SOLVING RIGHT TRIANGLES

Given a right triangle and some data about it, we say we've "solved" the triangle when we know the measures of its three sides and three angles. In the special case of a right triangle *ABC*, with *C* = 90° (see Figure T11), we are able to solve the triangle if we know either 1) the measure of two sides, or 2) the measures of one side and one additional angle.

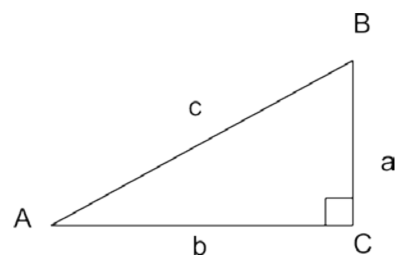


Figure T11

Example: Given triangle *ABC* with *C* = 90° and *a* = 2.6, *b* = 5.1, solve the triangle.

Solution: By the Pythagorean Theorem

$$c^2 = (2.6)^2 + (5.1)^2 = 32.77.$$

Hence, the hypotenuse $c = \sqrt{32.77} = 5.72$ (approximately)

Now we can calculate

$$\sin A = \frac{2.6}{\sqrt{32.77}} = 0.4542$$

so that *A* = 27°.

Then, *B* = 90° - 27° = 63°

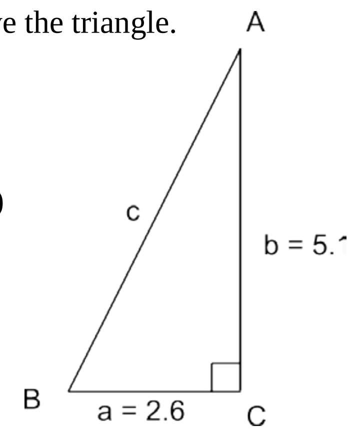


Figure T12

NOTE: All calculator work is rounded, so results are necessarily approximate.

Example: Given a right triangle *ABC* with $C = 90°$ and $a = 3.7$, $B = 54°$, solve the triangle.

Solution: Make a sketch (even if the proportions are wrong), see Figure T13.

Since

$$\cos B = \frac{3.7}{c} \text{ and } \tan B = \frac{b}{3.7},$$

we may find

$$c = \frac{3.7}{\cos B} = \frac{3.7}{\cos 54°} = 6.29$$

and

$$b = 3.7 \tan B = 3.7 \tan 54° = 5.09.$$

Finally,

$$A = 90° - 54° = 36°.$$

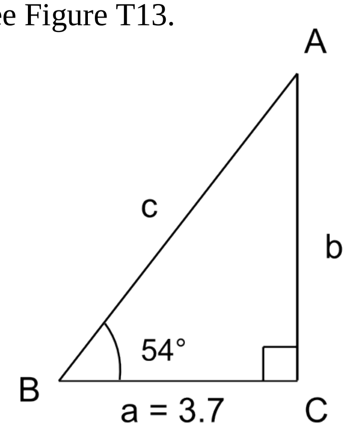


Figure T13

Problems:

17. What is the approximate perimeter of the triangle *ABC* where $a = 0.11$, $A = 21°$ and $C = 90°$?

18. In triangle *ABC*, $b = 65.39$, $c = 82.48$ and $C = 90°$. What is the approximate value of $A - B$?

19. In triangle *ABC*, $a = 2.63$, $b = 5.88$ and $C = 90°$. Find *c*, *A* and *B*.

SOLVING OBLIQUE TRIANGLES

A triangle which does not have a right (90°) angle is called an **oblique** triangle. An oblique triangle either has three acute (less than 90°) angles, or it has two acute angles and one obtuse (greater than 90°) angle. Figures T14 and T15 show the possibilities. These figures also show the standard labeling. The angles (or vertices) are denoted by capital letters and the sides opposite the angles are denoted by the same letters in lower cases. As with all triangles, the sum of the angles of an oblique triangle *ABC* is 180°,

$A + B + C = 180°$.

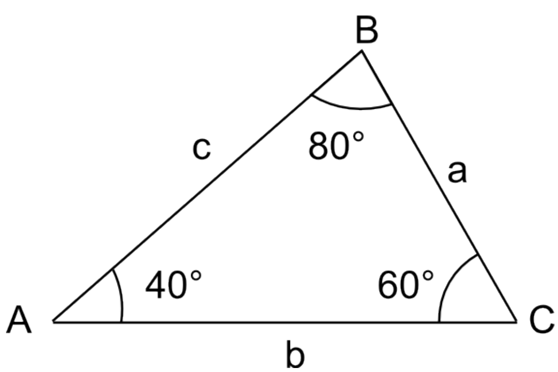


Figure T14

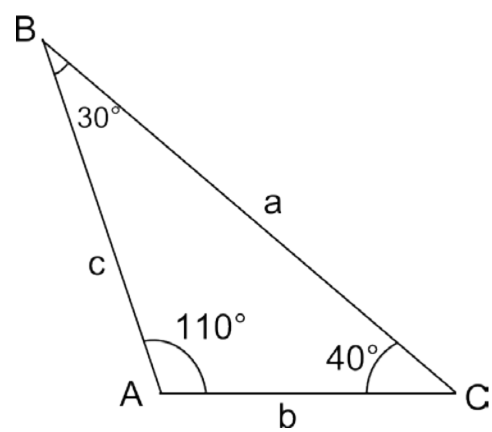


Figure T15

The equations relating sides and angles of right triangles are not true for oblique triangles. The basic relationships among sides and angles of an oblique triangle are given by the **Law of Sines** and the **Law of Cosines**. When three parts (including a side) of an oblique triangle are given, the Law of Sines or the Law of Cosines can be used to solve the triangle. There are four possible cases:

Case I: One side and two angles are given.

Case II: Two sides and one opposite angle are given.

Case III: Two sides and the included angle are given.

Case IV: Three sides are given.

The Law of Sines is used in Cases I and II. The Law of Cosines is used in Cases III and IV.

LAW OF SINES

For any triangle *ABC*,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

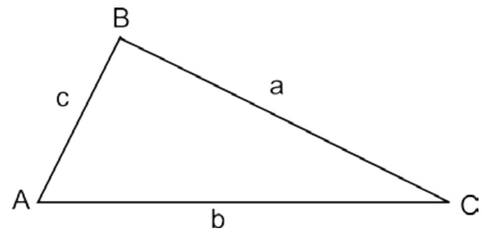


Figure T16

Example: Case I: One side and two angles given. Solve ΔBQK where $a = 12$, $Q = 110°$ and $K = 45°$.

Solution: In this case, the angle B is immediately found as

$$B = 180° - (Q + K) = 180° - 155° = 25°.$$

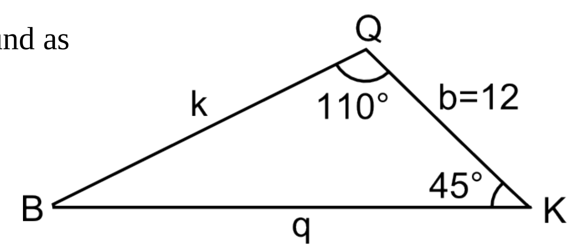


Figure T17

We can solve for q by the Laws of Sines:

$$\frac{q}{\sin 110°} = \frac{12}{\sin 25°}$$

so

$$q = \sin 110° \cdot \frac{12}{\sin 25°} = 26.68.$$

We can solve for k by the Law of Sines:

$$\frac{k}{\sin 45°} = \frac{12}{\sin 25°}$$

so

$$k = \sin 45° \cdot \frac{12}{\sin 25°} = 20.08.$$

Problems:

20. Find side b in ΔABC if $a = 11$, $A = 80$ and $B = 65°$.

21. Solve ΔPDQ where $q = 72.1$, $P = 27°$ and $Q = 25°$.

Case II: (Given two sides and one opposite angle) may also be solved by the Law of Sines — provided a solution exists. Case II is called the **ambiguous case** because there may be two solutions, or just one solution, or no solution at all. Use your review references to study the various possibilities. The figures on the next page show some of the possibilities for solutions (or no solution) given sides *a* and *c*, and angle *A*.

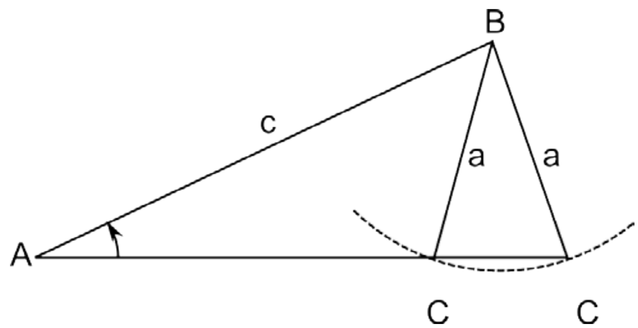


Figure T18 Two solutions

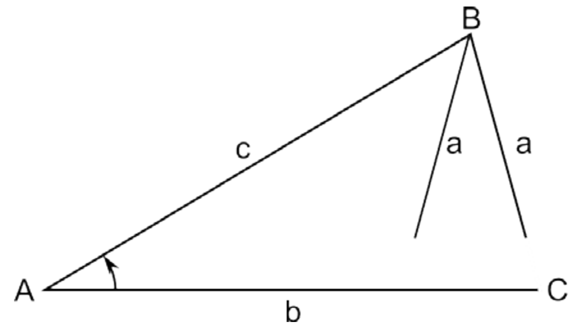


Figure T19 No solutions

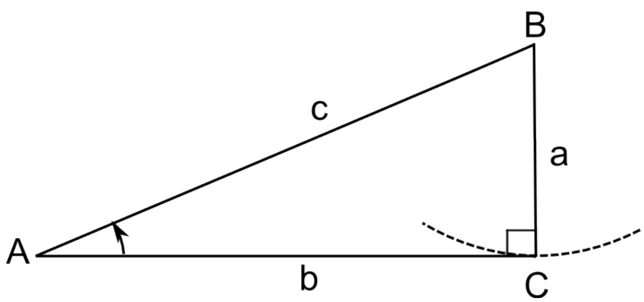


Figure T20 One solution

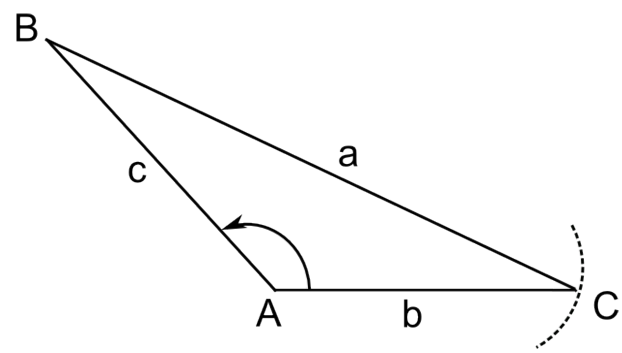


Figure T21 One solution

Example: Solve ΔABC where a = 10, c = 15 and A = 43°.

Solution: We attempt to solve for angle *C* by the Law of Sines:

$$\frac{15}{\sin C} = \frac{10}{\sin 43°}$$

so

$$\sin C = \frac{15 \cdot \sin 43°}{10} = 1.023.$$

There is no angle *C* for which sin *C* = 1.023. Hence, there is no triangle having these sides and angles.

Example: Solve Δ*ABC* where *a* = 8.4, *c* = 10.5 and *A* = 53.13°.

Solution: By the Law of Sines,

$$\frac{10.5}{\sin C} = \frac{8.4}{\sin 53.13} \text{ so } \sin C = \frac{10.5 \cdot \sin 53.13°}{8.4} = 1.000 \text{ (rounded).}$$

Hence *C* = 90°. Thus, in this case there is exactly one solution. To complete the solution, we find

$$B = 180° - (A + C) = 180° - (143.13°) = 36.87°.$$

Finally, solve for *b* by the Law of Sines:

$$\frac{b}{\sin 36.87°} = \frac{10.5}{\sin 90°} \text{ so } b = \frac{10.5 \sin 36.87°}{\sin 90°} = 6.30.$$

Example: Solve ΔABC where $a = 22$, $c = 27$ and $A = 50°$.

Solution: By the Law of Sines,

$$\frac{27}{\sin C} = \frac{22}{\sin 50°} \text{ so } \sin C = \frac{27 \sin 50°}{22} = 0.9401.$$

There are two angles C between 0° and 180° for which $\sin C = 0.9401$. They are $C_1 = 70.07°$ and $C_2 = 109.93°$. Since

$$A + C_1 = 50° + 70.07° = 120.07° < 180°$$

and

$$A + C_2 = 50° + 109.93° = 159.93° < 180°,$$

C_1 and C_2 each lead to a solution of the triangle. There are two triangles with given sides and angle. We must find them both.

When $C = C_1 = 70.07°$, $B = 180° - (70.07° + 50°) = 59.93°$. According to the Law of Sines, then

$$\frac{b}{\sin 59.93°} = \frac{22}{\sin 50°} \text{ so } b = \frac{22 \sin 59.93°}{\sin 50°} = 24.85.$$

The first solution to this triangle is

$a = 22$, $b = 24.85$, $c = 27$,

$A = 50°$, $B = 59.93°$, $C = 70.07°$.

When $C = C_2 = 109.93°$, $B = 180° - (109.93° + 50°) = 20.07°$.

Again by the Law of Sines,

$$\frac{b}{\sin 20.07°} = \frac{22}{\sin 50°} \text{ so } b = \frac{22 \sin 20.07°}{\sin 50°} = 9.86.$$

The second solution to this triangle is

$a = 22$, $b = 9.86$, $c = 27$,

$A = 50°$, $B = 20.07°$, $C = 109.93°$.

Example: Solve ΔABC where $a = 13$, $c = 6$ and $A = 70°$.

Solution: By the Law of Sines,

$$\frac{6}{\sin C} = \frac{13}{\sin 70°} \text{ so } \sin C = \frac{6 \sin 70°}{13} = 0.4337.$$

There are two angles C between 0° and 180° for which $\sin C = 0.4337$. They are $C_1 = 25.7°$ and $C_2 = 154.3°$. Since

$$A + C_1 = 70° + 25.7° = 95.7° < 180°,$$

C_1 can be an angle of a triangle which has the given sides and angle. Since

$$A + C_2 = 70° + 154.3° = 224.3° > 180°,$$

C_2 cannot be an angle of a triangle with the given sides and angle. This triangle has only one solution. It comes from $C_1 = 25.7°$.

When $C = C_1 = 25.7°$, $B = 180° - (70° + 25.7°) = 84.3°$. By the Law of Sines, then,

$$\frac{b}{\sin 84.3°} = \frac{13}{\sin 70°} \text{ so } b = \frac{13 \sin 84.3°}{\sin 70°} = 13.77.$$

The only triangle with sides and angle as given has

a = 13, b = 13.77, c = 6,

A = 70°, B = 84.3°, C = 25.7°.

Problems:

22. Solve ΔABC where $a = 221$, $c = 543$ and $A = 23°$.

23. Solve ΔABC where $a = 89.1$, $c = 100.0$ and $A = 63°$.

24. Solve ΔABC where $a = 9.2$, $c = 7.6$ and $A = 98.6°$.

25. In oblique triangle ABC, $a = 0.7$, $c = 2.4$ and $C = 98°$. Which one of the following might be angle A to the nearest tenth of a degree?

A) 3.4° **B)** 28.8° **C)** 73.2° **D)** 163.2° **E)** none of these

LAW OF COSINES

For any triangle ABC labeled in the customary way (as in Figure T22),

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

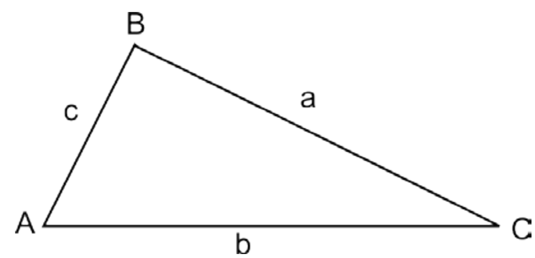


Figure T22

Notice that the Law of Cosines can also be stated as

$$b^2 = a^2 + c^2 - 2ac \cos B.$$

and

$$c^2 = a^2 + b^2 - 2ab \cos C.$$

When one angle of a triangle is 90°, the Law of Cosines reduces to the Pythagorean Theorem.

Example: Case III: Given two sides and the included angle. Solve ΔABC where $a = 7$, $b = 9$ and $C = 47°$.

Solution: By the Law of Cosines

$$c^2 = 7^2 + 9^2 - 2 \cdot 7 \cdot 9 \cos 47°.$$

Thus

$$c^2 = 49 + 81 - 85.9318 = 44.0682.$$

Hence

$$c = 6.64.$$

Now we may solve for angle A by the Law of Cosines.

$$a^2 = b^2 + c^2 - 2bc\cos A \text{ so } \cos A = \frac{9^2 + (6.64)^2 - 7^2}{(2)(9)(6.64)} = 0.6366.$$

Then $A = 50.46°$. Now we can find angle B as

$B = 180° - (50.46° + 47°) = 82.54°$.

The only triangle with given sides and angle has

$a = 7$, $b = 9$, $c = 6.64$,
$A = 50.46°$, $B = 82.54°$, $C = 47°$.

Example: Case IV: Given three sides. Solve ΔABC where $a = 6$, $b = 3$ and $c = 5$.

Solution: We begin by solving for angle A. (We could begin with any angle.)

$$a^2 = b^2 + c^2 - 2bc\cos A \text{ so } 6^2 = 3^2 + 5^2 - 2 \cdot 3 \cdot 5 \cos A$$

Then

$$\cos A = \frac{3^2 + 5^2 - 6^2}{2 \cdot 3 \cdot 5} = \frac{-2}{2 \cdot 3 \cdot 5} = -0.0666\ldots.$$

Hence,

$A = 93.82°$.

Next we solve for angle B by the Law of Cosines.

$$b^2 = a^2 + c^2 - 2ac\cos B \text{ so } 3^2 = 6^2 + 5^2 - 2 \cdot 6 \cdot 5 \cdot \cos B$$

Then

$$\cos B = \frac{6^2 + 5^2 - 3^2}{2 \cdot 6 \cdot 5} = 0.8666\ldots.$$

Hence,

$B = 29.93°$.

Now we can easily find angle C as

$C = 180° - (93.82° + 29.93°) = 56.25°$.

Thus the complete solution is

$a = 6$, $b = 3$, $c = 5$,
$A = 93.82°$, $B = 29.93°$, $C = 56.25°$.

Example: Case IV: Given three sides. Solve ΔABC where $a = 1$, $b = 4$ and $c = 2$.

Solution: When we apply the Law of Cosines, we find

$$1^2 = 4^2 + 2^2 - 2 \cdot 4 \cdot 2 \cos A$$

and

$$\cos A = \frac{4^2 + 2^2 - 1^2}{2 \cdot 4 \cdot 2} = \frac{19}{16} > 1.$$

There is no angle A with cosine greater than 1, so there is no triangle having these three sides. An attempt at $\cos B$ or $\cos C$ will also yield numbers outside the interval [-1 , 1]. One can also see that there is no solution by noticing that there are two sides ($a = 1$ and $c = 2$) whose sum is less than the third side, $1 + 2 < 4$.

Problems:

26. Solve ΔABC with $b = 33$, $c = 15$ and $A = 102°$.

27. Solve ΔABC with $a = 2$, $b = 3$ and $c = 7$.

28. Find angle R in ΔPQR where $p = 53$, $q = 101$ and $r = 95$.

29. Solve ΔPAS with $p = 7$, $a = 25$ and $s = 24$.

30. Solve ΔABC with $a = 7$, $b = 7$ and $c = 7$.

NARRATIVE PROBLEMS

In the following examples and problems it is important that you

1. *read* the problem carefully,
2. *draw* a reasonable sketch of the situation,
3. *identify* and *label* the known and the unknown quantities,
4. *write* an equation relating the known and unknown quantities,
5. *solve* the equation and
6. *examine* the reasonableness of your answer as it relates to the sketch.

Example: The Washington Monument casts a shadow 793 feet long when the angle of elevation of the sun is 35°. How tall is the monument? (Assume the ground is level, though you may know differently.)

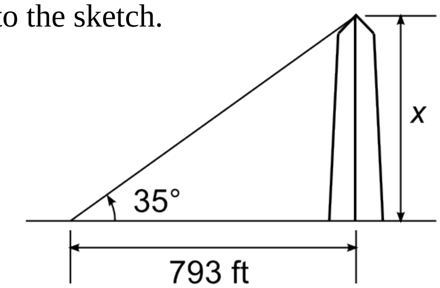


Figure T23

Solution: From the sketch as shown in Figure T23,

$$\tan 35° = \frac{x}{793}.$$

Hence

$$x = 793 \tan 35° = 555.26 \text{ feet}.$$

Example: Two men, 500 feet apart on level ground, observe a hot-air balloon between them. The respective angles of elevation are measured as 62.3° and 47.8°. What is the altitude of the balloon above the ground level?

Solution: Since $\frac{h}{a} = \sin C, \quad h = a \sin 47.8°$

We need to find side a. By the Law of Sines,

$$\frac{a}{\sin A} = \frac{b}{\sin B}.$$

Since $B = 180° - (62.3° + 47.8°) = 69.9°$, we get

$$\frac{a}{\sin 62.3°} = \frac{500}{\sin 69.9°}.$$

Thus,

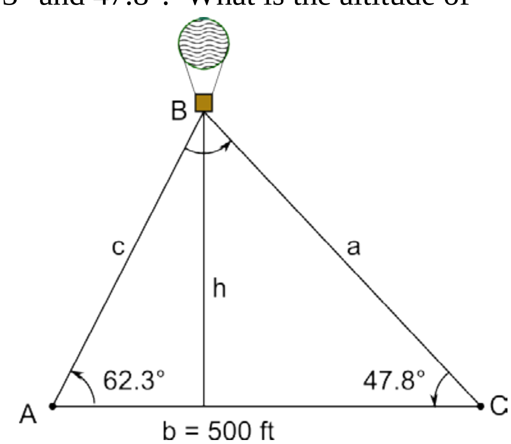


Figure T24

$$a = \frac{500 \sin 62.3°}{\sin 69.9°} = 471.41.$$

Hence,

$$h = a \sin 47.8° = (471.14) \sin 47.8° = 349.22 \text{ feet.}$$

Problems:

31. A surveyor determines the following information about a triangular shaped piece of property. One side has length 257.9 feet. A second side has length 185.1 feet. The angle opposite the 185.1 foot side measures 38.3°. How long is the third side of the property?

32. A merchant ship at its dock in New York was observed to subtend an angle of 6° from the window of an office 2000 feet from the bow and 3000 feet from the stern of the ship. How long is the ship?

33. A vertical tower is braced by two cables which are anchored to the ground at the same point. At this point, the angle between the cables is 15°. The first cable is 150 feet long and extends to the top of the tower. The second cable extends to a point 50 feet below the top of the tower. How long is the second cable?

34. As an airplane is taking off, its path of ascent makes a 60° angle with the runway. How many feet does the plane rise while it travels 1200 feet in the air? (Assume the path of the ascent is a straight line.)

GRAPHING TRIGONOMETRIC FUNCTIONS

Values of the trigonometric functions at 0°, 30°, 45°, 60° and 90° should be memorized. The following table gives the values.

degrees	radians	sine	cosine	tangent	cosecant	secant	cotangent
0°	0	0	1	0	undefined	1	undefined
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$
90°	$\frac{\pi}{2}$	1	0	undefined	1	undefined	0

Example: Using this table (not a calculator) find 2 sin 45° sec 45°.

Solution: From the table we have $\sin 45° = \frac{\sqrt{2}}{2}$ and $\sec 45° = \frac{1}{\cos 45°} = \sqrt{2}$. Hence,

$$2 \sin 45° \sec 45° = 2\frac{\sqrt{2}}{2} \cdot \sqrt{2} = 2.$$

Problems:

35. Evaluate $\csc^2 30° - \cot^2 30°$

36. Evaluate $2\cos\frac{\pi}{3}\sin\frac{\pi}{4}$.

37. Evaluate $\sec^2\frac{\pi}{3}+\tan^2\frac{\pi}{3}$.

38. Evaluate $\frac{2\tan 45°}{2-\sec^2 45°}$.

39. Which of the following is the exact numerical value of $\frac{\cot^2\frac{\pi}{3}-1}{2\cot\frac{\pi}{3}}$?

A) $\frac{\sqrt{3}}{2}$ **B)** $\frac{\sqrt{3}-3}{6}$ **C)** $\frac{\sqrt{3}}{3}$ **D)** $\frac{\sqrt{3}-1}{2}$ **E)** $\frac{-\sqrt{3}}{3}$

GRAPHS OF THE SINE AND COSINE FUNCTIONS

The graph of the function $y = \sin\theta$ is shown in Figure T25. The horizontal (or θ) axis is in radians, so the graph crosses the horizontal axis at 0, $\pm\pi$, $\pm 2\pi$, $\pm 3\pi$, etc. The maximum value of the sine function is 1, which occurs at $\pi/2$ and $\pi/2 \pm 2n\pi$ for each integer n. The minimum value of the sine function is -1. The section of the graph from $\theta = 0$ to $\theta = 2\pi$ is one complete cycle, *i.e.*, the graph repeats itself from 2π to 4π, from 4π to 6π, etc. Similarly, it repeats from -2π to 0, from -4π to -2π, etc.

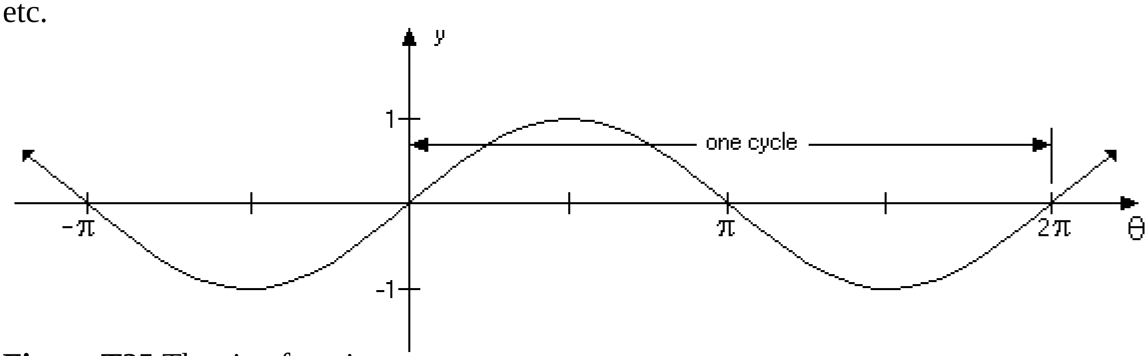


Figure T25 The sine function

The graph of the cosine function is shown in Figure T26. It, too, repeats in cycles of 2π and has values between -1 and 1. If the cosine graph is displaced $\pi/2$ units to the right, it coincides with sine graph, *i.e.*, $\cos(\theta - \pi/2) = \sin\theta$. Thus, both sine and cosine complete their cycle in 2π units.

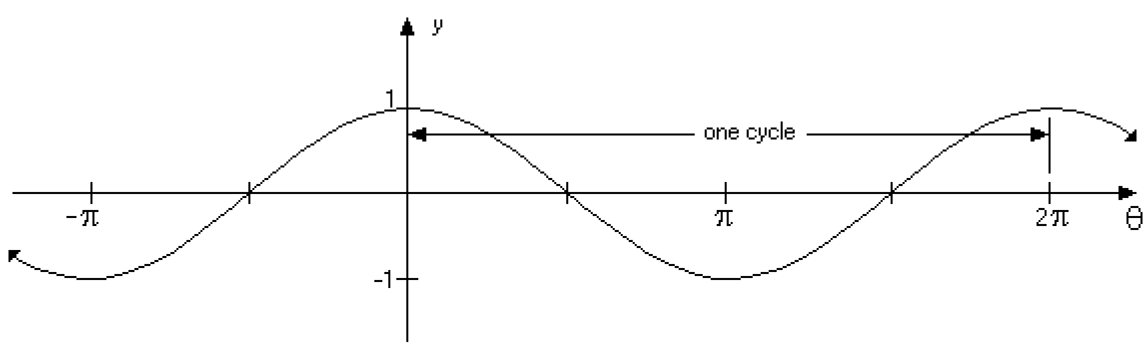


Figure T26 The cosine function

The function $y = 3 \sin \theta$ has the same cycle length as $y = \sin \theta$, namely 2π. but $y = 3 \sin \theta$ rises to maximum value of 3 (when $\theta = \pi/2$, for instance) and falls to a minimum of -3 (at $3\pi/2$, for instance). Its graph is shown below in Figure T27.

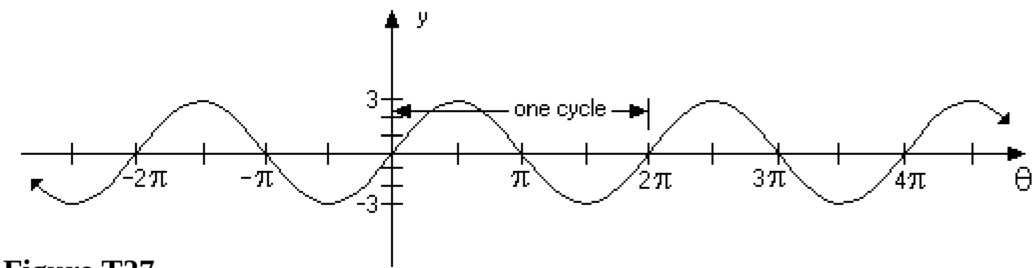


Figure T27

The maximum height above the horizontal axis of a graph that oscillates equally above and below the x-axis is called the **amplitude**. Thus, $y = \sin x$ has amplitude 1, $y = 2 \cos x$ has amplitude 2, and $y = -2 \sin x$ has amplitude 2.

Now consider the function $y = \sin 2\theta$. Its amplitude is 1, but because 2θ runs from 0 to 2π while θ is running from 0 to π, this function completes one cycle in π units. At a glance its graph looks exactly like the sine function. The only difference is that it cycles faster (twice as fast) as the ordinary sine function. Its graph is shown below in Figure T28.

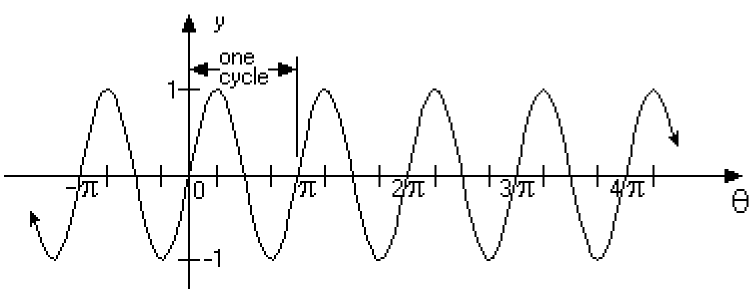


Figure T28

The **period** of this function is π, which is the length of one cycle. Now we combine these ideas. A function of the form $y = A \sin B\theta$ or $y = A \cos B\theta$ has

$$\text{Amplitude} = |A| \text{ and Period} = \frac{2\pi}{|B|}.$$

Problems:

40. Find the amplitude and period of the function $y = 3 \sin 2\theta$.

41. Find the amplitude and period of the function $y = -2 \cos 5\theta$.

42. Find the amplitude and period of the function $y = 4\cos\left(\frac{\theta}{3}\right)$.

43. Find the amplitude and period of the function shown in Figure T29. Find the equation for the function.

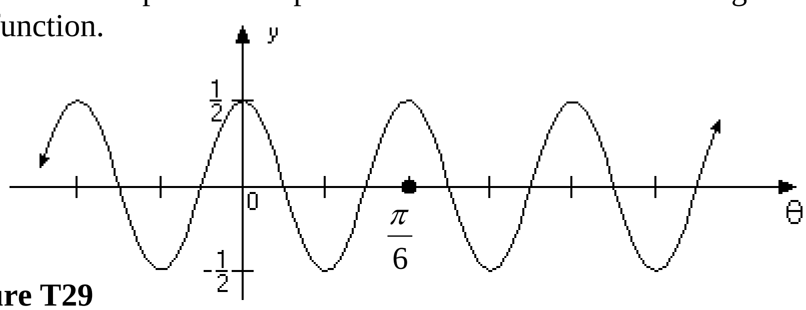


Figure T29

44. Find the amplitude and period of the function shown in Figure T30. Find the equation for the function.

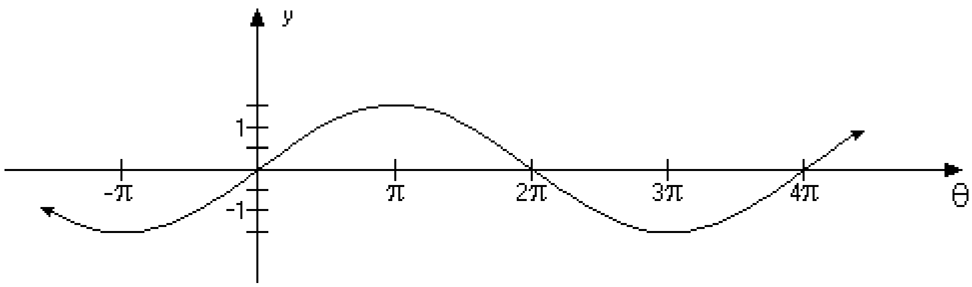
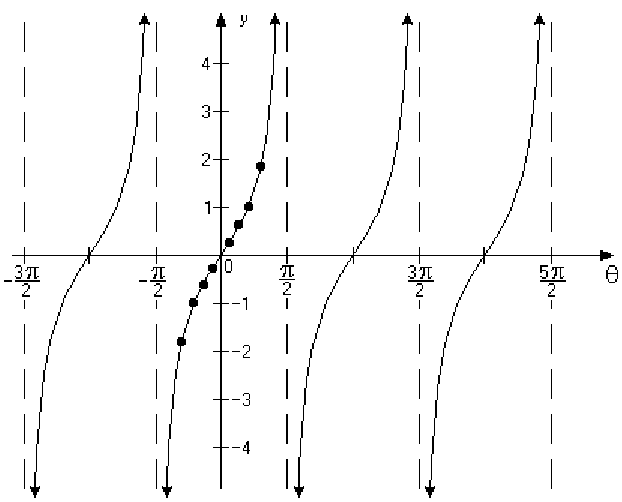


Figure T30

The graphs of the tangent and cotangent functions are shown in Figure T31 and Figure T32.



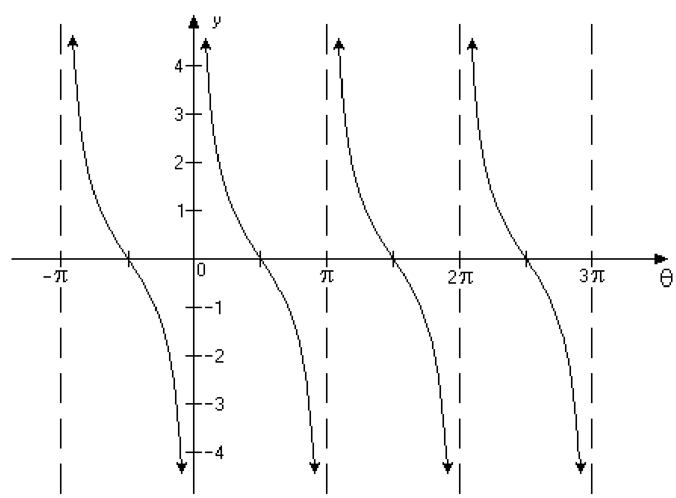


Figure T31 Graph of y = tan θ

Figure T32 Graph of y = cot θ

You should be able to construct these graphs. Note that both tangent and cotangent functions have period π rather than 2π. The tangent function has vertical asymptotes at odd multiples of $\pi/2$, while the cotangent function has asymptotes at multiples of π (that's the same as even multiples of $\pi/2$). Between two consecutive asymptotes, the tangent function is increasing while the cotangent function is decreasing.

The graph of the secant function appears in Figure T33.

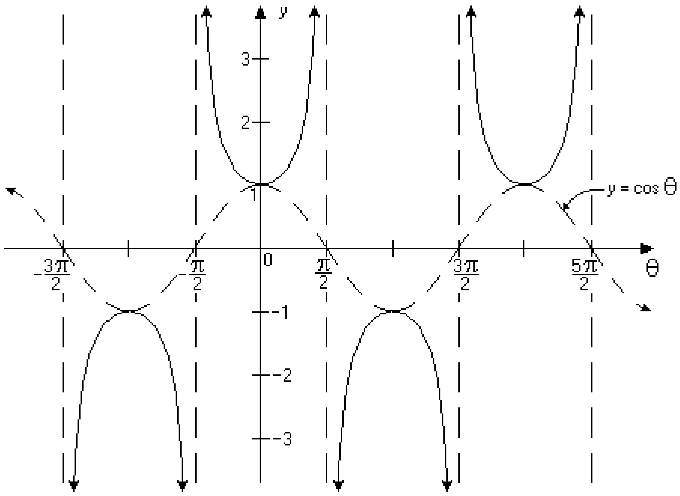


Figure T33 Graph of y = sec θ

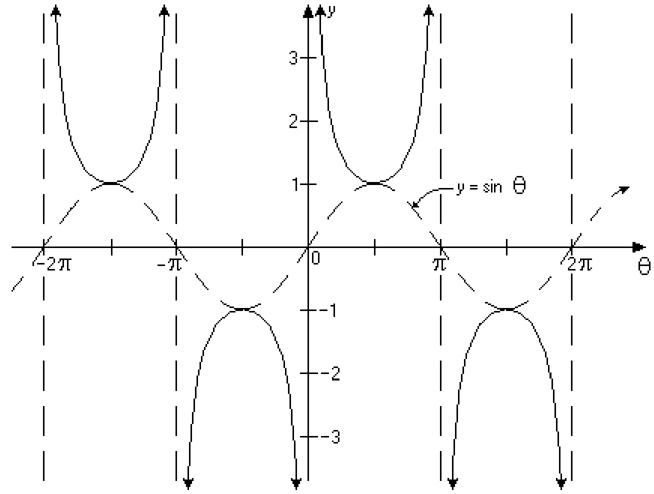


Figure T34 Graph of y = csc θ

The cosine function is also plotted so that you can see the reciprocal relationship $\sec\theta = 1/\cos\theta$ between these functions. Both the secant and cosecant have period 2π. For instance, one complete section of the graph of the secant lies between $\theta = -\pi/2$ and $\theta = 3\pi/2$. The secant function has vertical asymptotes at odd multiples of $\pi/2$ (where the cosine function is 0).

The graph of the cosecant function is shown in Figure T34. It is just the secant graph displaced $\pi/2$ units to the right. You should be able to construct the graphs of the secant and cosecant functions.

Problems:

45. The period of y = csc θ is ______________.

46. The period of y = tan θ is ______________.

47. The period of y = cos θ is ______________.

48. The period of y = cot θ is ______________.

49. The period of y = sin θ is ______________.

50. The period of y = sec θ is ______________.

51. Sketch the graph of the function y = cot θ by making a table of values. Use angles θ in the interval $0 \le \theta \le \pi$ which are multiples of $\pi/12$.

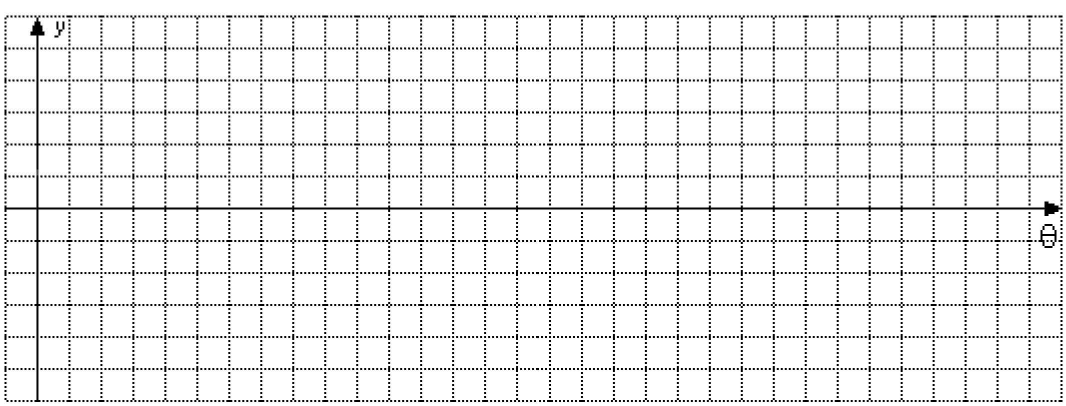


Figure T35a

52. Sketch the graph of the function $y = \csc\,\theta$ by making a table of values. Use angles θ in the interval $0 \le \theta \le 2\pi$ which are multiples of $\pi/12$.

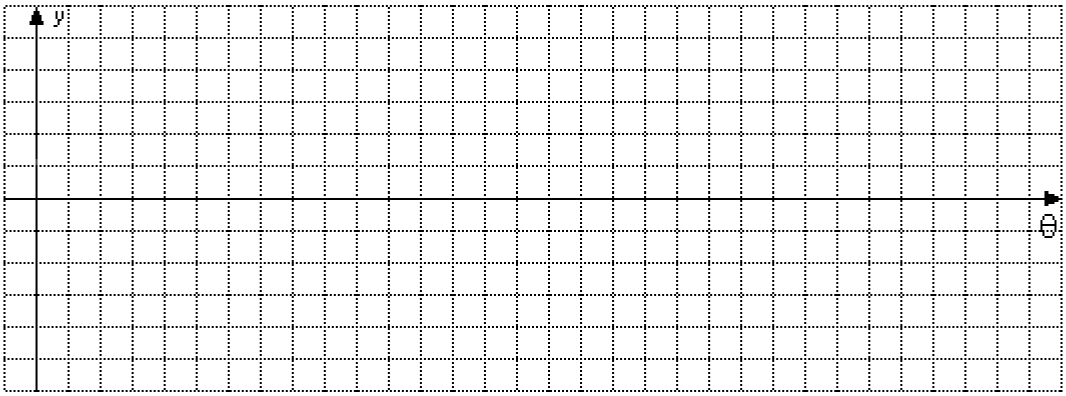


Figure T35b

53. Sketch the graph of the function $y = \sec\,\theta$ by making a table of values. Use angles θ in the interval $-\pi/2 \le \theta \le 3\pi/2$ which are multiples of $\pi/12$.

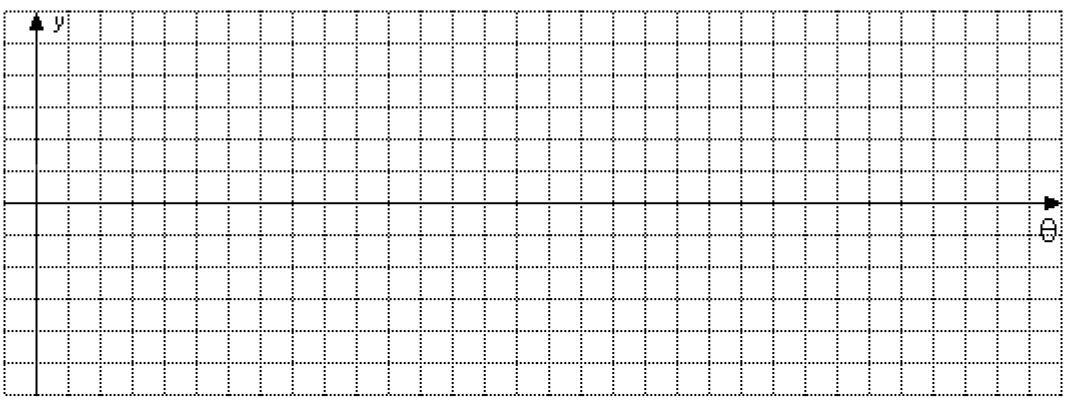


Figure T35c

INVERSE TRIGONOMETRIC FUNCTIONS

A portion of the graph of $y = \sin x$ is shown in Figure T36. Over the interval $x = -\pi/2$ to $x = \pi/2$, the sine function takes on each of its values exactly once. Consequently, for each y between -1 and +1 there is exactly one number x between $-\pi/2$ and $\pi/2$ for which $y = \sin x$. In this way the portion of the graph of $y = \sin x$ between $x = -\pi/2$ and $x = \pi/2$ determines a function which associates a unique number between $-\pi/2$ and $\pi/2$ with each number between -1 and 1. This function is called the inverse sine function and is denoted

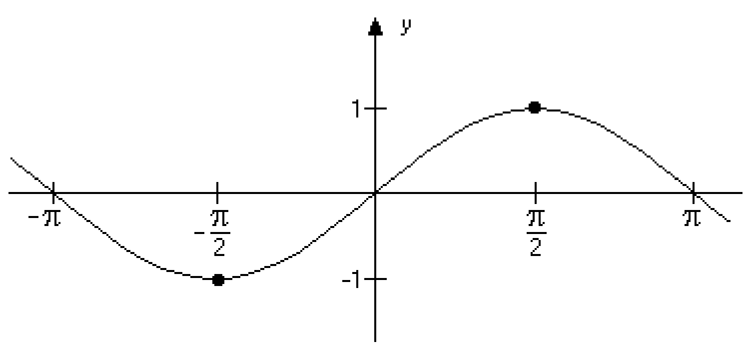


Figure T36 y = sin x

$$y = \sin^{-1}x.$$

Figure T37 shows the graph of $y = \sin^{-1}x$.

The **domain** of the inverse sine function is $-1 \le x \le 1$ and its **range** is $-\pi/2 \le y \le \pi/2$.

In this setting, when y is considered an angle, the unit of measure is radians, not degrees.

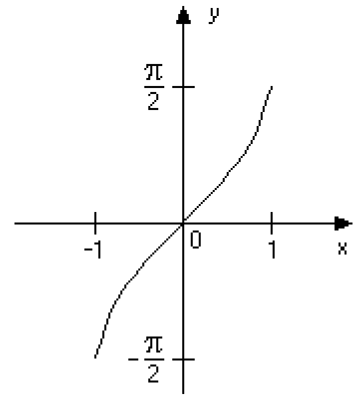


Figure T37 y = sin^{-1} x

The inverse cosine function $y = \cos^{-1}x$ and the inverse tangent function $y = \tan^{-1}x$ are defined in similar ways.

Figure T38 shows the graph of

$$y = \cos^{-1}x.$$

The **domain** of $y = \cos^{-1}x$ is $-1 \le x \le 1$ and its **range** is $0 \le y \le \pi$.

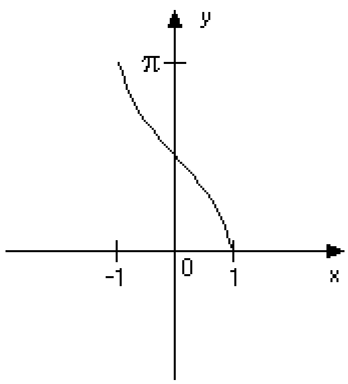


Figure T38 $y = \cos^{-1}x$.

Figure T39 shows the graph of

$$y = \tan^{-1}x.$$

The **domain** of the inverse tangent function is $-\infty < x < \infty$ and its **range** is $-\pi/2 < y < \pi/2$.

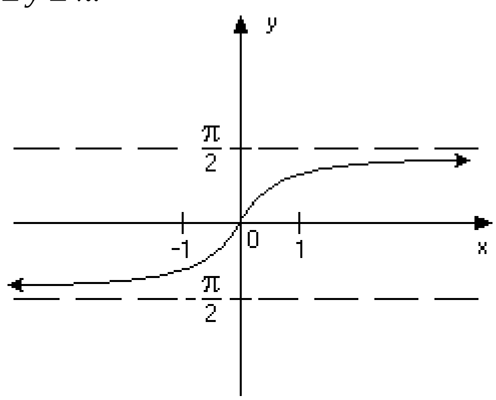


Figure T39 y = tan^{-1} x

In the functions $y = \sin^{-1}x$, $y = \cos^{-1}x$ and $y = \tan^{-1}x$, think of y as an angle expressed in radians. When using a scientific calculator to work with these functions, set the calculator to radian mode.

Example: Find $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$.

Solution: Set $y = \sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$. By the definition of the inverse sine function,

$$\sin y = \frac{\sqrt{2}}{2} \text{ and } -\frac{\pi}{2} \le y \le \frac{\pi}{2}.$$

The angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ whose sine is $\frac{\sqrt{2}}{2}$ is the familar special angle $\frac{\pi}{4}$. Thus,

$$\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}.$$

This function evaluation involved a familiar special angle, $\frac{\pi}{4}$, and so should be performed by inspection (as was done in the solution above). You can verify that the end result is correct by calculating either

$$\sin\left(\frac{\pi}{4}\right) \cong 0.7071067812 \cong \frac{\sqrt{2}}{2} \text{ or } \sin^{-1}\left(\frac{\sqrt{2}}{2}\right) \cong 0.7853981634 \cong \frac{\pi}{4}.$$

Example: Find $\tan^{-1}\left(-\sqrt{3}\right)$.

Solution: Set $y = \tan^{-1}\left(-\sqrt{3}\right)$. By the definition of the inverse tangent function,

$$\tan y = -\sqrt{3} \text{ and } -\frac{\pi}{2} < y < \frac{\pi}{2}.$$

Since tan *y* is negative, *y* must lie in the interval $-\frac{\pi}{2} < y < 0$. The angle between $-\frac{\pi}{2}$ and 0 whose tanget is $-\sqrt{3}$ is $-\frac{\pi}{3}$, so

$$\tan^{-1}\left(-\sqrt{3}\right) = -\frac{\pi}{3}.$$

Since this evaluation involved only a familiar special angle, it should be performed by inspection. Of course, the result can be verified by calculation.

Problems:
In Problems 53-56, find the value of the inverse function without using a calculator.

54. $\sin^{-1}\left(\frac{1}{2}\right)$

55. $\tan^{-1}\left(\frac{\sqrt{3}}{3}\right)$

56. $\cos^{-1}(-1)$

57. $\sin^{-1}(2)$

In Problems 57-59, find approximate values of the inverse trigonometric functions using a scientific calculator. Round off the final answer to 4 (four) decimal places.

58. $\cos^{-1}(-0.72)$

59. $\sin^{-1}(0.9901)$

60. $\tan^{-1}(-0.001)$

Many applications involve compositions of trigonometric and inverse trigonometric functions such as $\sin(\sin^{-1}x)$, $\sin^{-1}(\sin x)$ and $\sin(\cos^{-1}x)$. The simplest of these are of the form

$$\sin(\sin^{-1}x), \ \cos(\cos^{-1}x) \text{ and } \tan(\tan^{-1}x).$$

Consider the first of these. If $y = \sin^{-1}x$, then $x = \sin y$ and hence, $\sin(\sin^{-1}x) = \sin y = x$. Thus, as long as *x* is in the domain of the inverse sine function,

$$\sin(\sin^{-1}x) = x$$

Similarly, if $\cos^{-1}x$ is defined, then $\cos(\cos^{-1}x) = x$. If $\tan^{-1}x$ is defined, then $\tan(\tan^{-1}x) = x$.

Problems:

61. Evaluate $\cos(\cos^{-1}(-0.2119))$.

62. Evaluate $\sin(\sin^{-1}0)$.

63. Evaluate $\tan(\tan^{-1}(0.9531))$.

64. Evaluate $\sin(\sin^{-1}(-3.95))$.

Simplifying compositions such as $\sin^{-1}(\sin x)$, $\cos^{-1}(\cos x)$ and $\tan^{-1}(\tan x)$ requires more care.

Example: Evaluate $\tan^{-1}(\tan 1.07))$.

Solution: Since 1.07 lies in the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$, $\tan^{-1}(\tan 1.07) = 1.07$.

Example: Evaluate $\cos^{-1}\left(\cos\frac{8\pi}{7}\right)$.

Solution: The range of the inverse cosine function is the interval from 0 to π. Since $\frac{8\pi}{7} > \pi$, $\frac{8\pi}{7}$ is not in the range of the inverse cosine function. Consequently,

$$\cos^{-1}\left(\cos\frac{8\pi}{7}\right) \neq \frac{8\pi}{7}.$$

Our problem is to find an angle θ between 0 and π (in the range of $\cos^{-1}$) such that

$$\cos\frac{8\pi}{7} = \cos\theta.$$

A quadrant II angle having the same reference angle as $\frac{8\pi}{7}$ will meet this requirement. The reference angle for $\frac{8\pi}{7}$ is $\frac{\pi}{7}$. The quadrant II angle with reference angle $\frac{\pi}{7}$ is

$$\theta = \pi - \frac{\pi}{7} = \frac{6\pi}{7}.$$

Since $\cos\frac{8\pi}{7} = \cos\frac{6\pi}{7}$ and $\frac{6\pi}{7}$ is in the range of $\cos^{-1}$.

$$\cos^{-1}\left(\cos\frac{8\pi}{7}\right) = \cos^{-1}\left(\cos\frac{6\pi}{7}\right) = \frac{6\pi}{7}.$$

Verify this value using a scientific calculator.

Problems:

65. Evaluate $\tan(\tan^{-1}870)$.

66. Evaluate $\cos(\cos^{-1}5.7)$.

67. Evaluate $\tan^{-1}(\tan 1.4)$.

68. Evaluate $\tan^{-1}(\tan(-0.6217))$.

69. Evaluate $\sin^{-1}(\sin 0.0123)$.

70. Evaluate $\cos^{-1}\left(\cos\left(-\frac{\pi}{4}\right)\right)$.

71. Evaluate $\cos^{-1}(\cos(-2\pi))$.

72. Evaluate $\tan^{-1}(\tan\pi)$.

73. Evaluate $\tan^{-1}\left(\tan\dfrac{3\pi}{4}\right)$.

74. Evaluate $\sin^{-1}\left(\sin\dfrac{5\pi}{6}\right)$.

75. Evaluate $\sin^{-1}\left(\sin\dfrac{3\pi}{5}\right)$.

76. Evaluate $\cos^{-1}\left(\cos\dfrac{25\pi}{13}\right)$.

Finally, we evaluate composition such as $\sin(\cos^{-1}x)$ and $\tan(\sin^{-1}x)$ of a trigonometric function with some other inverse trigonometric function. These compositions cannot be evaluated directly from the definitions of the inverse trigonometric functions.

Example: Evaluate $\sin\left(\cos^{-1}\dfrac{4}{5}\right)$.

Solution: Set $y = \cos^{-1}\dfrac{4}{5}$. We must find $\sin y$. Since $\cos y = \dfrac{4}{5} > 0$, y is a quadrant I angle as shown in Figure T40. By the definition of the cosine function $\cos y = \dfrac{4}{5} = \dfrac{a}{r} = \dfrac{a}{\sqrt{a^2+b^2}}$ where (a , b) is any point on the terminal side of angle y. To find $\sin y$, we choose a point (a , b) so that

$$\frac{a}{\sqrt{a^2+b^2}} = \frac{4}{5}.$$

The easiest choice is $a = 4$, so

$$\sqrt{a^2+b^2} = \sqrt{16+b^2} = 5$$

$$16+b^2 = 25$$

$$b = \pm 3.$$

since the terminal side of the angle is in quadrant I, $b = +3$. The point $(a , b) = (4 , 3)$ and, by the definition of the sine function,

$$\sin\left(\cos^{-1}\frac{4}{5}\right) = \sin y = \frac{3}{5}.$$

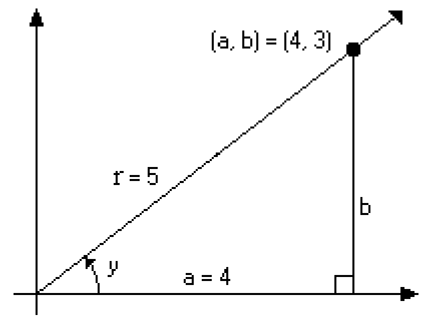


Figure T40

Example: Simplify $\cos\left(\tan^{-1}\left(-\dfrac{5}{2}\right)\right)$.

Solution: Set $y = \tan^{-1}\left(-\dfrac{5}{2}\right)$. Hence, $\tan y = -\dfrac{5}{2}$ and $-\dfrac{\pi}{2} < y < \dfrac{\pi}{2}$.

Since $\tan y$ is negative, y lies in the interval $-\dfrac{\pi}{2} < y < 0$ and y is a quadrant IV angle.

Sketch angle y in standard position in quadrant IV as shown in Figure T41.

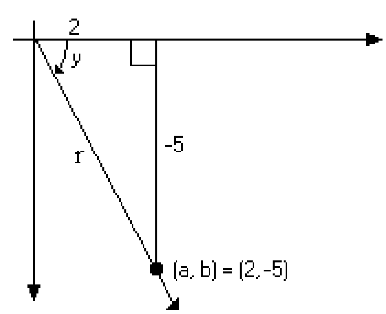


Figure T41

Choose a point (a , b) on the terminal side of y so that

$$\tan y = \frac{b}{a} = \frac{-5}{2}.$$

It is easiest to choose this point so $a = 2$ and $b = -5$.
The distance between the origin and the point (a , b) is

$$r = \sqrt{(-5)^2 + 2^2} = \sqrt{29}.$$

Hence, $\cos y = \dfrac{2}{\sqrt{29}}$ and $\cos\left(\tan^{-1}\left(-\dfrac{5}{2}\right)\right) = \dfrac{2}{\sqrt{29}}$.

Problems:

77. Evaluate $\cos\left(\sin^{-1}\dfrac{5}{13}\right)$.

78. Evaluate $\sin\left(\tan^{-1}\left(-\dfrac{5}{12}\right)\right)$.

79. Evaluate $\tan\left(\cos^{-1}\left(-\dfrac{3}{5}\right)\right)$.

80. Evaluate $\cos\left(\tan^{-1}1\right)$.

81. Evaluate $\sin\left(\cos^{-1}\dfrac{1}{2}\right)$.

82. Evaluate $\cos\left(\sin^{-1}(-1)\right)$.

83. Evaluate $\sin\left(\tan^{-1}2\right)$.

84. Evaluate $\tan\left(\cos^{-1}2\right)$.

85. Evaluate $\cos\left(\tan^{-1}\left(-\dfrac{8}{15}\right)\right)$.

86. Evaluate $\tan\left(\sin^{-1}\dfrac{15}{17}\right)$.

87. What is the exact value of $\sin\left(\tan^{-1}\dfrac{2\sqrt{11}}{5}\right)$?

A) $2\sqrt{11}$ **B)** $\dfrac{5}{\sqrt{19}}$ **C)** $\dfrac{\sqrt{19}}{5}$ **D)** $\dfrac{2\sqrt{11}}{\sqrt{69}}$ **E)** $\dfrac{5}{\sqrt{69}}$

BASIC TRIGONOMETRIC IDENTITIES

An **identity** is an equation that is true for all values of the variables for which each member of the equation is defined. An identity that relates trigonometric functions is called a **trigonometric identity**.

The **reciprocal identities**

$$\csc\theta = \frac{1}{\sin\theta}, \quad \sec\theta = \frac{1}{\cos\theta}, \quad \text{and} \quad \cot\theta = \frac{1}{\tan\theta}$$

and the **quotient identities**

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \quad \text{and} \quad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

are easily derived from the definition of the trigonometric functions of an angle θ in terms of a point (x , y) on the terminal side of the angle.

The variables x, y and r in the definitions of the trigonometric functions are related by the equation

$$x^2 + y^2 = r^2$$

so

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1.$$

Therefore, by the definitions of the sine and cosine functions,

$$\cos^2\theta + \sin^2\theta = 1.$$

Similarly,

$$1 + \tan^2\theta = \sec^2\theta$$

and

$$\cot^2\theta + 1 = \csc^2\theta.$$

These identities are called the **Pythagorean identities**.

The reciprocal, quotient and Pythagorean identities are used frequently. Memorize them or learn to construct them quickly by reasoning from the definitions as in the discussion above.

Problems:
Memorize the basic identities above. Then, without referring to the statements of the basic identities, fill in the blanks to form one of the reciprocal, quotient or Pythagorean identities, or one of their alternate forms.

88. $\tan^2\theta + 1 =$ ____________________

89. ____________________ $= \dfrac{\cos\theta}{\sin\theta}$

90. $\sin^2\theta + \cos^2\theta =$ ____________________

91. ____________________ $= \dfrac{1}{\cos\theta}$

92. ____________________ $= \dfrac{\sin\theta}{\cos\theta}$

93. $\csc^2\theta =$ ____________________

94. ____________________ $= \dfrac{1}{\tan\theta}$

95. $\cos^2\theta =$ ____________________

96. ____________________ $= \csc^2\theta - \cot^2\theta$

97. $\tan\theta\cot\theta =$ ____________________

98. ____________________ $= \dfrac{1}{\cot\theta}$

99. $\tan^2\theta - \sec^2\theta =$ ____________________

100. ____________________ $= 1 - \cos^2\theta$

101. $\sin\theta\cot\theta =$ ____________________

102. ____________________ $= \csc\theta$

103. ____________________ $= \dfrac{1}{\sec\theta}$

104. $\cos\theta\tan\theta =$ ____________________

105. ____________________ $= \dfrac{1}{\csc\theta}$

Given the quadrant of an angle and the value of one of its trigonometric functions, we can use the preceding identities to find the values of the other trigonometric functions.

Example: Find $\cot\theta$ if $\tan\theta = -5$.

Solution: Since $\cot\theta = \dfrac{1}{\tan\theta}$, we have $\cot\theta = \dfrac{1}{-5} = -\dfrac{1}{5}$.

Example: Find $\cos\theta$ if $\sin\theta = \dfrac{2}{5}$ and θ is a quadrant II angle.

Solution: Since $\cos^2\theta + \sin^2\theta = 1$,

$$\cos^2\theta + \left(\frac{2}{5}\right)^2 = 1 \text{ or } \cos^2\theta = 1 - \frac{4}{25} = \frac{21}{25}.$$

Then, $\cos\theta = -\sqrt{\dfrac{21}{25}} = -\dfrac{\sqrt{21}}{5}$, where the - sign is chosen because the cosine is negative in quadrant II.

Example: Find $\cot\theta$ if $\cos\theta = -\dfrac{1}{3}$ and θ is in quadrant III.

Solution: Since none of the basic identities involves just cot θ and cos θ, we first find sin θ and then find $\cot\theta$ as $\dfrac{\cos\theta}{\sin\theta}$. Now $\sin^2\theta + \left(-\dfrac{1}{3}\right)^2 = 1$.

So $\sin^2\theta = \dfrac{8}{9}$ and $\sin\theta = -\dfrac{2\sqrt{2}}{3}$. The negative sign is chosen because θ is in quadrant III. Finally,

$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{-1/3}{-2\sqrt{2}/3} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}.$$

Example: Find $\sin\theta$ if $\tan\theta = -\dfrac{1}{2}$ and θ is in quadrant IV.

Solution: First, $\cot\theta = \dfrac{1}{-1/2} = -2$.

Second, from the Pythagorean identity $\cot^2\theta + 1 = \csc^2\theta$, we get

$$(-2)^2 + 1 = \csc^2\theta$$

$$\csc^2\theta = 5.$$

Since θ is a quadrant IV angle, $\csc\theta = -\sqrt{5}$.

Finally, $\sin\theta = \dfrac{1}{\csc\theta} = \dfrac{1}{-\sqrt{5}} = -\dfrac{\sqrt{5}}{5}$.

Problems:

Use only the reciprocal identities, quotient identities and Pythagorean identities to solve Problems 106 – 113.

106. Find $\sec\theta$ if $\tan\theta = -4$ and θ is a quadrant IV angle.

107. Find $\csc\theta$ if $\sin\theta = \dfrac{2}{7}$.

108. Find $\cos\theta$ if $\sec\theta = -1$.

109. Find $\cot\theta$ if $\csc\theta = \dfrac{3}{2}$ and θ is a quadrant I angle.

110. Find $\tan\theta$ if $\cos\theta = -\dfrac{2}{3}$ and θ is a quadrant II angle.

111. Find $\cot\theta$ if $\sin\theta = -\frac{3}{5}$ and $\cos\theta = -\frac{4}{5}$.

112. Find $\sin\theta$ if $\sec\theta = \frac{7}{4}$ and θ is a quadrant IV angle.

113. Find $\sec\theta$ if $\csc\theta = 2$ and θ is a quadrant II angle.

Any trigonometric expression can be written entirely in terms of the sine and cosine functions by using the reciprocal and quotient identities.

Example: $\dfrac{\tan\theta}{\sec\theta} = \dfrac{\left(\sin\theta/\cos\theta\right)}{1/\cos\theta} = \dfrac{\sin\theta}{\cos\theta}\cdot\cos\theta = \sin\theta.$

Example: $\csc^2\theta + \sec^2\theta = \dfrac{1}{\sin^2\theta} + \dfrac{1}{\cos^2\theta} = \dfrac{\cos^2\theta + \sin^2\theta}{\sin^2\theta\cos^2\theta} = \dfrac{1}{\sin^2\theta\cos^2\theta}.$

Problems:
Write each expression in terms of the sine and cosine functions and then simplify the expression.

114. $\sec\theta + \tan\theta$

115. $\dfrac{\csc^2\theta - 1}{\cot\theta}$

116. $\dfrac{\sec\theta}{\tan\theta + \cot\theta}$

Here are some suggestions for verifying trigonometric identities successfully.

1. Become very familiar with the reciprocal, quotient and Pythagorean identities in all their forms.
2. Begin with the more complicated side of the identity and, using the basic identities, try to reduce it to the simpler side.
3. If necessary, convert all functions to the sine and cosine and then proceed to simplify each side.

Example: Verify the identity $2\sec^2\theta - 1 = \sec^4\theta - \tan^4\theta$.

Solution: Begin with the more complicated right-hand side and factor it.

$$\begin{aligned}\sec^4\theta - \tan^4\theta &= \left(\sec^2\theta + \tan^2\theta\right)\left(\sec^2\theta - \tan^2\theta\right)\\ &= \left(\sec^2\theta + \tan^2\theta\right)\cdot 1 = \sec^2\theta + \tan^2\theta\\ &= \sec^2\theta + \left(\sec^2\theta - 1\right) = 2\sec^2\theta - 1\end{aligned}$$

Example: Verify the identity $\tan\theta + \cot\theta = \csc\theta\sec\theta$.

Solution: Since no terms are squared, the Pythagorean identities are not helpful. Begin by writing the left side in terms of the sine and cosine.

$$\begin{aligned}\tan\theta + \cot\theta &= \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} = \frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta} = \frac{1}{\cos\theta\sin\theta}\\ &= \frac{1}{\sin\theta}\cdot\frac{1}{\cos\theta} = \csc\theta\cdot\sec\theta\end{aligned}$$

Problems:
Verify each of the following identities.

117. $\cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1$

118. $\cot^4\theta + 2\cot^2\theta + 1 = \csc^4\theta$

119. $(1 + \tan^2\theta) \cos^2\theta = 1$

120. $\sin^3\theta = \sin\theta - \sin\theta\cos^2\theta$

121. $\csc\theta(\cos\theta + \sin\theta) = \cot\theta + 1$

122. $\dfrac{1}{\sin\theta + 1} - \dfrac{1}{\sin\theta - 1} = 2\sec^2\theta$

123. $\dfrac{1 + \sin\theta}{\cos\theta} + \dfrac{\cos\theta}{\sin\theta - 1} = 0$

124. $\dfrac{1}{\cot\theta + \tan\theta} = \sin\theta\cos\theta$

125. $\sec^2\theta(1 + \sin\theta) = \dfrac{1}{1 - \sin\theta}$

126. $1 = \cos\theta\sin\theta(\tan\theta + \cot\theta)$

127. $\sec\theta - \tan\theta = \dfrac{1}{\sec\theta + \tan\theta}$

128. $\csc\theta + \cot\theta = \dfrac{1}{\csc\theta - \cot\theta}$

129. $\dfrac{\cos\theta + 1}{\tan^2\theta} = \dfrac{\cos\theta}{\sec\theta - 1}$

130. $\cos^6\theta = 1 - 3\sin^2\theta + 3\sin^4\theta - \sin^6\theta$

NEGATIVE-ANGLE, SUM, DIFFERENCE AND COFUNCTION IDENTITIES

If θ is an angle in standard position and (x, y) is a point on its terminal side, then $(x, -y)$ is a point on the terminal side of $-\theta$. Also, (x, y) and $(x, -y)$ are the same distance r from the origin. Consequently,

$$\sin(-\theta) = \frac{-y}{r} = -\sin\theta,$$

$$\cos(-\theta) = \frac{x}{r} = \cos\theta$$

and $$\tan(-\theta) = \frac{-y}{x} = -\tan\theta.$$

The trigonometric identities

$$\sin(-\theta) = -\sin\theta,\ \cos(-\theta) = \cos\theta \text{ and } \tan(-\theta) = -\tan\theta$$

are called the **negative-angle identities**. They should be memorized.

The difference identity for the cosine function says that for any two angles θ and ϕ

$$\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi.$$

The derivation of this identity involves the distance formula and the Law of Cosines and is somewhat intricate. Read it in your favorite trigonometry reference. To help you remember the difference identity for the cosine, remember that when $\theta = \phi$ this identity reduces to the Pythagorean identity

$\cos^2\theta + \sin^2\theta = 1$.

By replacing ϕ by $-\phi$ in the difference identity for the cosine and then using the negative angle identities, we obtain the sum identity for the cosine:

$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi.$$

By considering quadrant I angles (especially angles smaller than $\pi/4$ radians) one can see that if θ is an angle in standard position and (x, y) is a point on its terminal side, then (y, x) is a point on the terminal side of $\pi/2 - \theta$. Also, (x, y) and (y, x) are the same distance from the origin. From the definition of the trigonometric functions of an angle in terms of a point on the terminal side of the angle, it follows that for any angle θ

$$\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta \qquad \cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta$$

$$\tan\left(\frac{\pi}{2}-\theta\right)=\cot\theta \qquad \cot\left(\frac{\pi}{2}-\theta\right)=\tan\theta$$

These relationships, called the **cofunction identities**, are also useful in degree form.

$$\sin(90° - \theta) = \cos\theta \qquad \cos(90° - \theta) = \sin\theta$$

$$\tan(90° - \theta) = \cot\theta \qquad \cot(90° - \theta) = \tan\theta$$

Problems:
Memorize the negative-angle identities, the difference and sum identities for the cosine function, and the cofunction identities. Then, without referring to the statements of these identities, fill in the missing members of the following identities.

131. $\cos(\theta + \phi) =$ ____________________

132. ____________________ $= -\tan\theta$

133. $\sin(90° - \theta) =$ ____________________

134. $\cos(-\theta) =$ ____________________

135. ____________________ $= \cos\theta\cos\phi + \sin\theta\sin\phi$

136. $\cot\left(\frac{\pi}{2}-\theta\right) =$ ____________________

137. $\sin(-\theta) =$ ____________________

138. $\cos\theta\cos\phi - \sin\theta\sin\phi =$ ____________________

139. ____________________ $= -\sin\theta$

140. $\cos\left(\frac{\pi}{2}-\theta\right) =$ ____________________

141. $\tan(90° - \theta) =$ ____________________

142. $\cos(\theta - \phi) =$ ____________________

143. $\tan(-\theta) =$ ____________________

We can also use these identities to find values of the trigonometric functions for certain special angles.

Example: Find cos (150°).

Solution: $\cos(150°)=\sin(90°-150°)=\sin(-60°)=-\sin 60°=-\frac{\sqrt{3}}{2}$

Example: Find $\cos\left(\theta+\frac{\pi}{3}\right)$, if $\sin\theta=\frac{3}{\sqrt{10}}$ and $\cos\theta=\frac{1}{\sqrt{10}}$.

Solution:

$$\cos\left(\theta+\frac{\pi}{3}\right)=\cos\theta\cos\frac{\pi}{3}-\sin\theta\sin\frac{\pi}{3}$$

$$=\frac{1}{\sqrt{10}}\cdot\frac{1}{2}-\frac{3}{\sqrt{10}}\cdot\frac{\sqrt{3}}{2}$$

$$=\frac{\sqrt{10}-3\sqrt{30}}{20}$$

Problems:

In Problems 144-146, evaluate each expression using the negative-angle identities.

144. $\tan\left(-\frac{\pi}{6}\right) =$

145. cos (90°) + sin (-180°) =

146. $\sin\left(-\frac{\pi}{3}\right)\tan\left(-\frac{4\pi}{3}\right) =$

In Problems 147-149, evaluate each expression using the cofunction identities.

147. $\cos\frac{\pi}{3} =$

148. $\cot\frac{3\pi}{4} =$

149. tan (90° - 45°) + cos (90° - 180°) =

In Problems 150-154, use the sum and difference identities for the cosine function to evaluate each expression.

150. $\cos\frac{\pi}{12} = \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$

151. cos 105°

152. $\cos\frac{7\pi}{12}\cos\frac{5\pi}{12} - \sin\frac{7\pi}{12}\sin\frac{5\pi}{12}$

153. $\cos(30° - \theta)$ if $\sin\theta = \frac{1}{\sqrt{5}}$ and $\cos\theta = -\frac{2}{\sqrt{5}}$

154. $\cos\left(\theta + \frac{\pi}{4}\right)$ if $\sin\theta = -\frac{\sqrt{2}}{2}$ and $\cos\theta = \frac{\sqrt{2}}{2}$

By replacing θ with $\frac{\pi}{2}$ - θ in the difference and sum identities for the cosine and then applying the cofunction identities, we obtain the sum and difference identities for the sine function:

$$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$

and

$$\sin(\theta - \phi) = \sin\theta\cos\phi - \cos\theta\sin\phi.$$

The sum and difference identities for the tangent function are developed from the sum and difference identities for the sine and cosine and the quotient identities. These identities state that for any angles θ and ϕ

$$\tan(\theta + \phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}$$

and

$$\tan(\theta - \phi) = \frac{\tan\theta - \tan\phi}{1 + \tan\theta\tan\phi}$$

You should either memorize all of these identities or learn to reconstruct them quickly by reasoning similar to that in the discussion.

Example: Verify the identity $\cos(180° - \theta) = -\cos\theta$.

Solution:

$$\cos(180° - \theta) = \cos 180° \cos\theta + \sin 180° \sin\theta$$
$$= -1 \cdot \cos\theta + 0 \cdot \sin\theta$$
$$= -\cos\theta$$

Example: Verify the identity $\dfrac{\cos(\theta + \phi)}{\sin\theta\cos\phi} = \cot\theta - \tan\phi$.

Solution: We apply the identity to the quantity $\cos(\theta + \phi)$ and then simplify the fractions.

$$\frac{\cos(\theta + \phi)}{\sin\theta\cos\phi} = \frac{\cos\theta\cos\phi - \sin\theta\sin\phi}{\sin\theta\cos\phi} = \frac{\cos\theta\cos\phi}{\sin\theta\cos\phi} - \frac{\sin\theta\sin\phi}{\sin\theta\cos\phi}$$
$$= \frac{\cos\theta}{\sin\theta} - \frac{\sin\phi}{\cos\phi} = \cot\theta - \tan\phi$$

Problems:
Verify each of the following identities.

155. $\cos(\theta + \pi) = -\cos\theta$

156. $\cos(\theta + \phi) + \cos(\theta - \phi) = 2\cos\theta\cos\phi$

157. $\tan^2(-\theta) - \sec^2\theta = -1$

158. $\dfrac{\cos(\theta - \phi) - \cos(\theta + \phi)}{\cos(\theta - \phi) + \cos(\theta + \phi)} = \tan\theta\tan\phi$

DOUBLE-ANGLE AND HALF-ANGLE IDENTITIES

The double-angle identities are obtained by setting $\theta = \phi$ in the three sum identities:

$$\sin 2\theta = \sin(\theta + \theta) = \sin\theta\cos\theta + \cos\theta\sin\theta = 2\sin\theta\cos\theta,$$

$$\cos 2\theta = \cos(\theta + \theta) = \cos\theta\cos\theta - \sin\theta\sin\theta = \cos^2\theta - \sin^2\theta$$

and

$$\tan(2\theta) = \tan(\theta + \theta) = \frac{\tan\theta + \tan\theta}{1 - \tan\theta\tan\theta}.$$

Thus, the **double-angle identities** are:

$$\sin 2\theta = 2\sin\theta\cos\theta,$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

and

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}.$$

The **half-angle identities** are derived from the **double-angle identities** for the cosine. From the Pythagorean identity $\cos^2\theta = 1 - \sin^2\theta$, so

$$\cos 2\phi = \cos^2\phi - \sin^2\phi = 1 - 2\sin^2\phi.$$

By solving for $\sin\phi$, we obtain

$$\sin\phi = \pm\sqrt{\frac{1-\cos 2\phi}{2}}.$$

Finally, by replacing ϕ with $\frac{\theta}{2}$, we obtain

$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{2}},$$

where the sign is determined from the quadrant of $\frac{\theta}{2}$. By a similar calculation

$$\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos\theta}{2}},$$

where the sign is determined by the quadrant of $\frac{\theta}{2}$.

Finally, to obtain a formula for $\tan\frac{\theta}{2}$, use the half-angle identities for the sine and cosine in

$$\tan\left(\frac{\theta}{2}\right) = \frac{\sin\left(\theta/2\right)}{\cos\left(\theta/2\right)}$$

and get

$$\tan\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}},$$

where, again, the sign is determined by the quadrant of $\frac{\theta}{2}$.

You should memorize the double-angle and half-angle identities by learning to reconstruct them quickly from previous identities as in the discussion above.

Problems:

159. $\sin 2\theta =$ ____________________

160. ____________ $= \sin\theta\cos\phi - \cos\theta\sin\phi$

161. $1 - 2\sin^2\theta =$ ____________________

162. $\tan(\theta + \phi) =$ ____________________

163. ____________________ $= \pm\sqrt{\frac{1+\cos\theta}{2}}$

164. ____________________ $= \sin(\theta + \phi)$

165. $\cos 2\theta =$ ____________________

166. $\pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} =$ ____________________

167. ____________________ $= 2\cos^2\theta - 1$

168. $\sin\frac{\theta}{2} =$ ____________________

169. ____________________ $= \frac{\tan\theta - \tan\phi}{1+\tan\theta\tan\phi}$

170. $\sin(\theta - \phi) =$ ____________________

171. $\tan 2\theta =$ ____________________

172. ____________________ $= \cos^2\theta - \sin^2\theta$

173. $2\sin\theta\cos\theta =$ ____________________

174. $\sin\theta\cos\phi + \cos\theta\sin\phi =$ ____________

175. ________________________ $= \dfrac{2\tan\theta}{1-\tan^2\theta}$

The half-angle, double-angle, and sum and difference identities can be used to evaluate the trigonometric functions for certain special angles. The following examples illustrate this use.

Example: Evaluate sin 75°

Solution: Since 75° = 30° + 45°, we have
sin 75° = sin (30° + 45°) = sin 30° cos 45° + cos 30° sin 45°

$$= \frac{1}{2}\cdot\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{2}+\sqrt{6}}{4}$$

Example: Evaluate $\cos\frac{\pi}{8}$.

Solution: Apply the half-angle identity for the cosine to the angle $\frac{\pi}{8} = \frac{1}{2}\left(\frac{\pi}{4}\right)$.

$$\cos\frac{\pi}{8} = \cos\left(\frac{1}{2}\left(\frac{\pi}{4}\right)\right) = \sqrt{\frac{1+\cos\frac{\pi}{4}}{2}} = \sqrt{\frac{1+\sqrt{2}/2}{2}} = \frac{1}{2}\sqrt{2+\sqrt{2}}.$$

Notice that the + sign is chosen for the radical because $\frac{\pi}{8}$ is a quadrant I angle and the cosine is positive in quadrant I.

Example: Find sin 2θ, if $\sin\theta = -\frac{8}{17}$ and θ is a quadrant IV angle.

Solution: By the double-angle identity,

$$\sin 2\theta = 2\sin\theta\cos\theta.$$

Since $\sin\theta = -\frac{8}{17}$, the missing piece in this puzzle is the value of $\cos\theta$. Since $\sin^2\theta + \cos^2\theta = 1$, we have

$$\left(-\frac{8}{17}\right)^2 + \cos^2\theta = 1$$

so that

$$\cos^2\theta = 1 - \frac{64}{289} = \frac{225}{289}$$

and hence,

$$\cos\theta = \pm\frac{15}{17}.$$

We must choose the + sign because θ is a quadrant IV angle and the cosine is positive in quadrant IV. Thus,

$$\cos\theta = \frac{15}{17}$$

and

$$\sin 2\theta = 2\left(-\frac{8}{17}\right)\left(\frac{15}{17}\right) = -\frac{240}{289}.$$

Problems:

Use trigonometric identities to evaluate each expression.

176. $\sin\frac{\pi}{12} = \sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$

177. $\cos^2 165° - \sin^2 165°$

178. $2\sin\frac{\pi}{12}\cos\frac{\pi}{12}$

179. Find $\sin\frac{\theta}{2}$ if $\cos\theta = \frac{3}{4}$ and θ is a quadrant I angle.

180. Find $\cos 2\theta$ if $\cos\theta = \frac{3}{4}$.

181. Find $\sin(\theta + \phi)$ if $\sin\theta = \frac{3}{5}$, $\sin\phi = \frac{12}{13}$ and both θ and ϕ are quadrant I angles.

182. Find $\tan(\theta + \phi)$ if $\tan\theta = \frac{1}{3}$ and $\tan\phi = \frac{2}{3}$.

We may use the recently developed identities to verify more complicated trigonometric identities.

Example: Verify the identity $\sin(\theta + \phi) \cdot \sin(\theta - \phi) = \sin^2\theta - \sin^2\phi$.

Solution: From the sum and difference formulas for the sine, we have

$$\begin{aligned}
\sin(\theta+\phi)\cdot\sin(\theta-\phi) &= (\sin\theta\cos\phi + \cos\theta\sin\phi)(\sin\theta\cos\phi - \cos\theta\sin\phi) \\
&= \sin^2\theta\cos^2\phi - \cos^2\theta\sin^2\phi \\
&= \sin^2\theta(1-\sin^2\phi) - (1-\sin^2\theta)\sin^2\phi \\
&= \sin^2\theta - \sin^2\theta\sin^2\phi - \sin^2\phi + \sin^2\theta\sin^2\phi \\
&= \sin^2\theta - \sin^2\phi.
\end{aligned}$$

Example: Verify the identity $\sec 2\theta = \frac{\sec^2\theta}{2 - \sec^2\theta}$.

Solution: $$\sec 2\theta = \frac{1}{\cos 2\theta} = \frac{1}{2\cos^2\theta - 1} = \frac{1}{2\left(\frac{1}{\sec^2\theta}\right) - 1} = \frac{\sec^2\theta}{2 - \sec^2\theta}$$

Problems:

In the following problems, verify that each equation is a trigonometric identity.

183. $\cot(\theta + \phi) = \frac{\cot\theta\cot\phi - 1}{\cot\theta + \cot\phi}$

184. $\cot 2\theta = \frac{\cot^2\theta - 1}{2\cot\theta}$

185. $\frac{(\sin\theta + \cos\theta)^2 - 1}{\sin 2\theta} = 1$

186. $\frac{\sin 2\theta}{2\sin\theta} = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}$

187. $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$ (HINT: $3\theta = \theta + 2\theta$)

TRIGONOMETRIC EQUATIONS

Most of the trigonometric equations we have seen up to this point have been identities; *i.e.*, they were true for all values of the variables. Now we wish to consider **conditional equations —** ones which are true for some, but not all, values of the variables. To solve a conditional equation means to find all values of the variables for which the equality holds.

Example: The trigonometric equation $\sin\theta = 1$ is true for some values of θ and not for others. It is not true for $\theta = \pi/6$, neither is it true for θ = your social security number. There are an infinite number of solutions to this equation. They are given by

$$\theta = \frac{\pi}{2} \pm 2\pi n, \quad \text{where } n \text{ is an integer.}$$

Some trigonometric equations don't have any solutions; *i.e.*, they are not true for any of the variables.

Example: Solve $\sin\theta = 3 - \cos\theta$.

Solution: Rewrite this equation as $\sin\theta + \cos\theta = 3$. For every value of θ, $\sin\theta + \cos\theta < 2$. Therefore, there are no values of θ for which $\sin\theta + \cos\theta = 3$. Hence, this equation has no solutions.

There is no general procedure for solving all trigonometric equations and some can be very difficult. For example, you might try your hand at the innocent-looking $\tan\theta = \theta$. (After you run out of good ideas with identities and algebraic manipulations, you might get out your calculator and try some approximations.)

We will be concerned only with equations that can be solved by using trigonometric identities to change the form of the equation and applying algebraic methods such as rearranging terms, factoring, squaring, and taking roots.

The simplest trigonometric equations are

$$\sin\theta = c, \quad \cos\theta = c \quad \text{and} \quad \tan\theta = c,$$

where c is a constant. Inverse trigonometric functions can be used to find one solution to equations of this form, when the equation has a solution. Our problem is to use this information to find all solutions.

The equations $\sin\theta = c$ and $\cos\theta = c$ have two solutions in the interval $0 \le \theta < 2\pi$ when $|c| < 1$, and one solution when $|c| = 1$. Since the sine and cosine functions have period 2π, to find all solutions, add and subtract integer multiples of 2π to the solutions in the interval $0 \le \theta < 2\pi$.

For any number c, the equation $\tan\theta = c$ has one solution in the interval $0 \le \theta < \pi$. Since the tangent function has period π, to find all solutions, add integer multiples of π to the solution in the interval $0 \le \theta < \pi$.

An expression (or expressions) which describes all solutions to a trigonometric equation in a simple way is called the **general solution** to the equation.

Example: Solve the trigonometric equation $\cos\theta = \frac{1}{2}$.

Solution: One solution is

$$\theta_1 = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}.$$

The second solution in the interval $0 \le \theta < 2\pi$ is

$$\theta_2 = 2\pi - \theta_1 = \frac{5\pi}{3}.$$

The only solutions in the interval $0 \le \theta < 2\pi$ are $\theta_1 = \frac{\pi}{3}$ and $\theta_2 = \frac{5\pi}{3}$. We use these angles to represent the general solution:

$\theta = \frac{\pi}{3} + 2n\pi$ or $\theta = \frac{5\pi}{3} + 2n\pi$, where n is any integer.

All of these solutions can be represented in a single expression as

$\theta = \pm\frac{\pi}{3} + 2n\pi$, n an integer.

Example: Find the general solution of the equation $\tan\theta = \frac{-\sqrt{3}}{3}$.

Solution: One solution is

$$\theta_1 = \mathrm{Tan}^{-1}\left(\frac{-\sqrt{3}}{3}\right) = \frac{-\pi}{6}.$$

The general solution can be represented as

$$\theta = \frac{-\pi}{6} + n\pi, \quad n \text{ an integer.}$$

However, it is customary to represent the general solution in terms of a solution between 0 and π. The angle θ_1 is not in this interval, so we use $\theta_1 + \pi = \frac{5\pi}{6}$ to represent the general solution as

$$\theta = \frac{5\pi}{6} + n\pi, \quad n \text{ an integer.}$$

Problems:

In problems 188-191, find the general solution to each trigonometric equation.

188. $\sin\theta = \frac{\sqrt{2}}{2}$

189. $\tan\theta = -1$

190. $\cos\theta = 0$

191. $\sin\theta = -\frac{\sqrt{3}}{2}$

Equations of the form

$$\sin B\theta = c, \quad \cos B\theta = c \quad \text{and} \quad \tan B\theta = c,$$

where B and c are constants, are slightly more complicated than the equations just considered. Since $\sin B\theta$ and $\cos B\theta$ have period $\frac{2\pi}{B}$, to express the general solution to $\sin B\theta = c$ and $\cos B\theta = c$, add (positive and negative) integer multiples of $\frac{2\pi}{B}$ to the solutions in the interval $0 \le \theta < \frac{2\pi}{B}$. The function $\tan B\theta$ has period $\frac{\pi}{B}$ so the general solution to $\tan B\theta = c$ is expressed by adding integer multiples of $\frac{\pi}{B}$ to the solutions in the interval $0 \le \theta < \frac{\pi}{B}$.

The technique for solving these equations is to transform them into the form of the simpler equations just considered by making a change of variables.

Example: Find all solutions to $\sin 2\theta = \frac{\sqrt{2}}{2}$ in the interval $0 \le \theta < 2\pi$.

Solution: Make the change of variables $\phi = 2\theta$ so the equation becomes

$$\sin\phi = \frac{\sqrt{2}}{2}.$$

The general solution to the original equation is

$$\phi = \frac{\pi}{4} + 2n\pi, \quad \phi = \frac{3\pi}{4} + 2n\pi, \ n \text{ an integer.}$$

Thus, the general solution to the original equation is

$$2\theta = \frac{\pi}{4} + 2n\pi, \quad 2\theta = \frac{3\pi}{4} + 2n\pi, \ n \text{ an integer.}$$

or

$$\theta = \frac{\pi}{8} + n\pi, \quad \theta = \frac{3\pi}{8} + n\pi, \ n \text{ an integer.}$$

The solutions in the interval $0 \le \theta < 2\pi$ are

$$\theta = \frac{\pi}{8}, \ \frac{3\pi}{8}, \ \frac{9\pi}{8}, \ \frac{11\pi}{8}.$$

Problems:

In problems 192-195, find all solutions in the interval $0 \le \theta < 2\pi$.

192. $\cos 2\theta = -\frac{1}{2}$

193. $\tan 3\theta = 0$

194. $\sin\frac{\theta}{2} = -\frac{1}{2}$

195. $\sin 4\theta = -\frac{\sqrt{3}}{2}$

We now extend our techniques to solve equations that yield to rearranging and factoring.

Example: Find all solutions of $\cos\theta\tan\theta = \cos\theta$ in $0 \le \theta < 2\pi$.

Solution: Rewrite $\cos\theta\tan\theta = \cos\theta$ as

$$\cos\theta\tan\theta - \cos\theta = 0$$

and factor to obtain

$$\cos\theta(\tan\theta - 1) = 0.$$

Either $\cos\theta = 0$ or $\tan\theta - 1 = 0$.

We solve these equations separately in the stated interval.

$\cos\theta = 0$ gives solutions $\theta = \frac{\pi}{2}, \frac{3\pi}{2}.$

$\tan\theta - 1 = 0$ (or $\tan\theta = 1$) gives solutions $\frac{\pi}{4}$ and $\frac{5\pi}{4}.$

Thus, the solutions in the interval $0 \le \theta < 2\pi$ are

$$\theta = \frac{\pi}{2}, \frac{3\pi}{2} \quad \text{and } \theta = \frac{\pi}{4}, \frac{5\pi}{4}.$$

Example: Find all solutions of $2\sin^2\theta + 1 = 3\sin\theta$ in $0 \le \theta < 2\pi$.

Solution: Rewrite the equation as

$$2\sin^2\theta - 3\sin\theta + 1 = 0$$

and factor as a quadratic in $\sin\theta$ to obtain

$$(2\sin\theta - 1)(\sin\theta - 1) = 0.$$

Set each factor equal to 0 and solve the two equations

$$2\sin\theta - 1 = 0 \text{ and } \sin\theta - 1 = 0$$

in the interval $0 \le \theta < 2\pi$. The first yields solutions $\frac{\pi}{6}$ and $\frac{5\pi}{6}$, while the second yields $\theta = \frac{\pi}{2}$. The solutions in the interval $0 \le \theta < 2\pi$ are

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6} \text{ and } \theta = \frac{\pi}{2}.$$

Problems:

In problems 196-200, find all solutions in the interval $0 \le \theta < 2\pi$

196. $4\sin^2\theta = 1$

197. $2\cos^2\theta - \cos\theta = 1$

198. $4\cos^4\theta = 3 - \cos^2\theta$

199. $\sin\theta\cos\theta + \cos\theta = 1 + \sin\theta$

200. $\sin^2 2\theta = \sin 2\theta$

There are two situations in which identities are commonly used to solve trigonometric equations.

1. When the equation involves more than one trigonometric function, an identity may be used to rewrite the equation in terms of just one function.
2. When the trigonometric equation involves an unknown angle θ and its multiples, we may use the double-angle, half-angle and addition formulas to rewrite the equation in terms of just one angle.

These techniques are illustrated in the following examples.

Example: Find all solutions of $\sec^2\theta + \tan\theta = 1$ in $0 \le \theta < 2\pi$.

Solution: We use the identity $\sec^2\theta = 1 + \tan^2\theta$ to rewrite the equation as

$$1 + \tan^2\theta + \tan\theta = 1.$$

Subtract 1 from both sides and factor to get

$$\tan\theta(\tan\theta + 1) = 0.$$

Thus $\tan\theta = 0$ or $\tan\theta + 1 = 0$, so the solutions in $0 \le \theta < 2\pi$ are

$$\theta = 0, \ \pi \quad \text{and} \quad \theta = \frac{3\pi}{4}, \frac{7\pi}{4}.$$

Example: Find all solutions of $\sin\theta - \cos 2\theta = 0$ in the interval $0 \le \theta < 2\pi$.

Solution: We use the identity $\cos 2\theta = 1 - 2\sin^2\theta$ to rewrite the equation as

$$\sin\theta - (1 - 2\sin^2\theta) = 0$$

which is now in terms of θ (and not 2θ). Solving by factoring,

$$2\sin^2\theta + \sin\theta - 1 = 0$$
$$(\sin\theta + 1)(2\sin\theta - 1) = 0$$
$$\sin\theta = -1 \quad \text{or} \quad \sin\theta = \frac{1}{2}.$$

The solutions in the interval $0 \le \theta < 2\pi$ are

$$\theta = \frac{3\pi}{2} \quad \text{and} \quad \theta = \frac{\pi}{6}, \quad \frac{5\pi}{6}.$$

Example: Find all solutions of $\sin\theta + \cos\theta = 1$ in the interval $0 \le \theta < 2\pi$.

Solution: Rewrite the equation as

$$\cos\theta = 1 - \sin\theta.$$

Square both sides to obtain

$$\cos^2\theta = 1 - 2\sin\theta + \sin^2\theta.$$

Use the identity $\cos^2\theta = 1 - \sin^2\theta$ to obtain

$$1 - \sin^2\theta = 1 - 2\sin\theta + \sin^2\theta.$$

Now solve by factoring.

$$2\sin^2\theta - 2\sin\theta = 0$$
$$\sin\theta(\sin\theta - 1) = 0$$
$$\sin\theta = 0 \quad \text{or} \quad \sin\theta = 1.$$

The solutions in the interval $0 \le \theta < 2\pi$ are

$$\theta = 0, \quad \pi \quad \text{and} \quad \theta = \frac{\pi}{2}.$$

Finally, the squaring process in the first step may have introduced extraneous roots. We check the values $\theta = 0, \frac{\pi}{2}, \pi$ in the original equation to see if they really are solutions.

$\theta = 0$: $\sin 0 + \cos 0 = 0 + 1 = 1$ $\quad$ $\theta = 0$ is a solution

$\theta = \frac{\pi}{2}$: $\sin\frac{\pi}{2} + \cos\frac{\pi}{2} = 1 + 0 = 1$ $\quad$ $\theta = \frac{\pi}{2}$ is a solution

$\theta = \pi$: $\sin\pi + \cos\pi = 0 + (-1) = -1 \ne 1$ $\quad$ $\theta = \pi$ is not a solution

Thus, the solutions are $\theta = 0, \quad \theta = \frac{\pi}{2}.$

Problems:

In problems 200-206, find all solutions of the given equation in the interval $0 \le \theta < 2\pi$.

201. $\cot 2\theta = -1$

202. $4\sin\theta\cos\theta = -\sqrt{3}$

203. $2\cos\theta\cos 2\theta - 2\sin\theta\sin 2\theta = 1$

204. $\cos^2\theta = \sin^2\theta + 1$

205. $4 + 2\cot^2\theta = 5\csc\theta$

206. $\cos\theta = \cos^2\frac{\theta}{2}$

207. $1 + \tan\theta = \sec\theta$

II. TRIGONOMETRY – ANSWER SECTION

ANGLES

1. -120° **2.** -765°, $-\frac{17\pi}{4}$ radians **3.** $\frac{3\pi}{5}$ radians **4.** 70° **5.** -52.9°

ARC LENGTH

6. 4.76 **7.** 22.62

AREA OF A SECTOR

8. 61.58 **9.** 10.47

THE TRIGONOMETRIC FUNCTIONS

10. (a) $\frac{\sqrt{77}}{9}$; (b) 14.62 **11.** -2.71624 **12.** 6.68153

13. tan θ is undefined **14.** -4.12838 **15.** -5.73769 **16. C)** 309°

SOLVING RIGHT TRIANGLES

17. 0.70 **18.** -14.9° **19.** c = 6.44, A = 24.1°, B = 65.9°

SOLVING OBLIQUE TRIANGLES

20. b = 10.12

21. D = 128°, d = 134.4, p = 77.5

22. 1) C = 73.7°, B = 83.3°, b = 561.7
2) C = 106.3°, B = 50.7°, b = 437.7

23. no solution

24. C = 54.8°, B = 26.6°, b = 4.2

25. E) none of these ($A \cong 16.79°$)

LAW OF COSINES

26. a = 38.99, B = 55.87°, C = 22.13° **27.** no solution (2 + 3 < 7) **28.** 68.15°

29. P = 16.26°, A =90°, S = 73.74° ($25^2 = 24^2 + 7^2$) **30.** $A = B = C = 60°$

NARRATIVE PROBLEMS

31. 295.7 feet **32.** 1,032.3 feet **33.** 113.3 feet **34.** 1,039.2 feet

GRAPHING TRIGONOMETRIC FUNCTIONS

35. 1 **36.** $\frac{\sqrt{2}}{2}$ **37.** 7 **38.** undefined **39. E)** $-\frac{\sqrt{3}}{3}$

GRAPHS OF THE SINE AND COSINE FUNCTIONS

40. amplitude = 3; period = π

41. amplitude = 2; period = $\frac{2\pi}{5}$

42. amplitude = 4; period = 6π

43. amplitude = $\frac{1}{2}$; period = $\frac{\pi}{6}$; $y = \frac{1}{2}\cos(12\theta)$

44. amplitude = $\frac{3}{2}$; period = 4π; $y = \frac{3}{2}\sin\left(\frac{\theta}{2}\right)$

45. 2π **46.** π **47.** 2π **48.** π **49.** 2π **50.** 2π

51. see page 16, Figure T32 **52.** see page 17, Figure T34 **53.** see page 17, Figure T33

INVERSE TRIGONOMETRIC FUNCTIONS

54. $\frac{\pi}{6}$ **55.** $\frac{\pi}{6}$ **56.** π **57.** undefined (2 is not in the domain of $\text{Sin}^{-1}x$)

58. 2.3746 **59.** 1.4300 **60.** -0.0010 **61.** -0.2119 **62.** 0 **63.** 0.9531

64. no solution **65.** 870 **66.** undefined **67.** 1.4 **68.** -0.6217 **69.** 0.0123

70. $\frac{\pi}{4}$ **71.** 0 **72.** 0 **73.** $-\frac{\pi}{4}$ **74.** $\frac{\pi}{6}$ **75.** $\frac{2}{5}\pi$

76. $\frac{\pi}{13}$ **77.** $\frac{12}{13}$ **78.** $-\frac{5}{13}$ **79.** $-\frac{4}{3}$ **80.** $\frac{\sqrt{2}}{2}$ **81.** $\frac{\sqrt{3}}{2}$

82. 0 **83.** $\frac{2}{5}\sqrt{5}$ **84.** undefined **85.** $\frac{15}{17}$ **86.** $\frac{15}{8}$ **87. D)** $\frac{2\sqrt{11}}{\sqrt{69}}$

BASIC TRIGONOMETRIC IDENTITIES

88. $\sec^2\theta$ **89.** $\cot\theta$ **90.** 1 **91.** $\sec\theta$ **92.** $\tan\theta$ **93.** $1 + \cot^2\theta$

94. $\cot\theta$ **95.** $1 - \sin^2\theta$ **96.** 1 **97.** 1 **98.** $\tan\theta$ **99.** -1

100. $\sin^2\theta$ **101.** $\cos\theta$ **102.** $\frac{1}{\sin\theta}$ **103.** $\cos\theta$ **104.** $\sin\theta$ **105.** $\sin\theta$

106. $\sqrt{17}$ **107.** $\frac{7}{2}$ **108.** -1 **109.** $\frac{\sqrt{5}}{2}$ **110.** $-\frac{\sqrt{5}}{2}$ **111.** $\frac{4}{3}$

112. $-\frac{\sqrt{33}}{7}$ **113.** $-\frac{2}{3}\sqrt{3}$ **114.** $\frac{1+\sin\theta}{\cos\theta}$ **115.** $\cot\theta$ **116.** $\sin\theta$

117. $\cos^2\theta - \sin^2\theta = \cos^2\theta - (1 - \cos^2\theta) = 2\cos^2\theta - 1$

118. $\cot^4\theta + 2\cot^2\theta + 1 = (\cot^2\theta + 1)^2 = (\csc^2\theta)^2 = \csc^4\theta$

119. $\left(1+\tan^2\theta\right)\cos^2\theta = \left(1+\dfrac{\sin^2\theta}{\cos^2\theta}\right)\cos^2\theta = \cos^2\theta + \sin^2\theta = 1$

120. $\sin\theta - \sin\theta\cos^2\theta = \sin\theta(1 - \cos^2\theta) = \sin^3\theta$

121. $\csc\theta(\cos\theta + \sin\theta) = \dfrac{1}{\sin\theta}\left(\cos\theta + \sin\theta\right) = \dfrac{\cos\theta}{\sin\theta} + 1 = \cot\theta + 1$

122. $\dfrac{1}{\sin\theta+1} - \dfrac{1}{\sin\theta-1} = \dfrac{\left(\sin\theta-1\right)-\left(\sin\theta+1\right)}{\sin^2\theta-1} = \dfrac{-2}{-\cos^2\theta} = 2\sec^2\theta$

123. $\dfrac{1+\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{\sin\theta-1} = \dfrac{\sin^2\theta-1+\cos^2\theta}{\cos\theta\left(\sin\theta-1\right)} = \dfrac{1-1}{\cos\theta\left(\sin\theta-1\right)} = 0$

124. $\dfrac{1}{\cot\theta+\tan\theta} = \dfrac{1}{\dfrac{\cos\theta}{\sin\theta}+\dfrac{\sin\theta}{\cos\theta}} = \dfrac{1}{\dfrac{\left(\cos^2\theta+\sin^2\theta\right)}{\cos\theta\sin\theta}} = \dfrac{1}{\dfrac{1}{\cos\theta\sin\theta}} = \cos\theta\sin\theta$

125. $\sec^2\theta\left(1+\sin\theta\right) = \sec^2\theta\cdot\dfrac{1+\sin\theta}{1}\cdot\dfrac{1-\sin\theta}{1-\sin\theta} = \dfrac{\sec^2\theta\left(1-\sin^2\theta\right)}{1-\sin\theta} = \dfrac{\sec^2\theta\cos^2\theta}{1-\sin\theta} = \dfrac{1}{1-\sin\theta}$

126. $\cos\theta\sin\theta\left(\tan\theta+\cot\theta\right) = \cos\theta\sin\theta\left(\dfrac{\sin\theta}{\cos\theta}+\dfrac{\cos\theta}{\sin\theta}\right) = \sin^2\theta+\cos^2\theta = 1$

127. $\dfrac{1}{\sec\theta+\tan\theta} = \dfrac{1}{\sec\theta+\tan\theta}\cdot\dfrac{\sec\theta-\tan\theta}{\sec\theta-\tan\theta} = \dfrac{\sec\theta-\tan\theta}{\sec^2\theta-\tan^2\theta} = \sec\theta-\tan\theta$

128. $\csc\theta+\cot\theta = \dfrac{\left(\csc\theta+\cot\theta\right)}{1}\cdot\dfrac{\csc\theta-\cot\theta}{\csc\theta-\cot\theta} = \dfrac{\csc^2\theta-\cot^2\theta}{\csc\theta-\cot\theta} = \dfrac{1}{\csc\theta-\cot\theta}$

129. $\dfrac{\cos\theta+1}{\tan^2\theta} = \dfrac{\cos\theta+\cos\theta\sec\theta}{\sec^2\theta-1} = \dfrac{\cos\theta(1+\sec\theta)}{(\sec\theta+1)(\sec\theta-1)} = \dfrac{\cos\theta}{\sec\theta-1}$

130. $\cos^6\theta = (1 - \sin^2\theta)^3 = (1 - \sin^2\theta)(1 - 2\sin^2\theta + \sin^4\theta) = 1 - 3\sin^2\theta + 3\sin^4\theta - \sin^6\theta$

NEGATIVE-ANGLE, SUM, DIFFERENECE AND CO-FUNCTION IDENTITIES

131. $\cos\theta\cos\phi - \sin\theta\sin\phi$

132. $\tan(-\theta)$

133. $\cos\theta$

134. $\cos\theta$

135. $\cos(\theta - \phi)$

136. $\tan\theta$

137. $-\sin\theta$

138. $\cos(\theta + \phi)$

139. $\sin(-\theta)$

140. $\sin\theta$

141. $\cot\theta$

142. $\cos\theta\cos\phi + \sin\theta\sin\phi$

143. $-\tan\theta$

144. $-\dfrac{\sqrt{3}}{3}$

145. 0

146. $\dfrac{3}{2}$

147. $\dfrac{1}{2}$

148. -1

149. 1

150. $\dfrac{\sqrt{2}+\sqrt{6}}{4}$

151. $\dfrac{\sqrt{2}-\sqrt{6}}{4}$

152. -1

153. $\dfrac{\sqrt{5}-2\sqrt{15}}{10}$

154. 1

155. cos ($\theta + \pi$) = cosθ cosπ - sinθ sinπ = -cosθ - 0 = -cosθ

156. cos ($\theta + \phi$) + cos ($\theta - \phi$) = cosθ cosϕ - sinθ sinϕ + cosθ cosϕ + sinθ sinϕ = 2cosθ cosϕ

157. $\tan^2(-\theta) - \sec^2\theta = (\tan(-\theta))^2 - \sec^2\theta = (-\tan\theta)^2 - \sec^2\theta = \tan^2\theta - \sec^2\theta = -1$

158. $$\frac{\cos(\theta-\phi)-\cos(\theta+\phi)}{\cos(\theta-\phi)+\cos(\theta+\phi)} = \frac{(\cos\theta\cos\phi+\sin\theta\sin\phi)-(\cos\theta\cos\phi-\sin\theta\sin\phi)}{(\cos\theta\cos\phi+\sin\theta\sin\phi)+(\cos\theta\cos\phi-\sin\theta\sin\phi)}$$
$$= \frac{2\sin\theta\sin\phi}{2\cos\theta\cos\phi} = \tan\theta\tan\phi$$

DOUBLE-ANGLE AND HALF-ANGLE IDENTITIES

159. 2sinθ cosθ

160. sin ($\theta - \phi$)

161. cos 2θ

162. $\frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}$

163. $\cos\frac{\theta}{2}$

164. sinθ cosϕ + cosθ sinϕ

165. $\cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$

166. $\tan\frac{\theta}{2}$

167. cos 2θ

168. $\pm\sqrt{\frac{1-\cos\theta}{2}}$

169. tan ($\theta - \phi$)

170. sinθ cosϕ - cosθ sinϕ

171. $\frac{2\tan\theta}{1-\tan^2\theta}$

172. cos 2θ

173. sin 2θ

174. sin ($\theta + \phi$)

175. tan 2θ

176. $\frac{\sqrt{6}-\sqrt{2}}{4}$

177. $\frac{\sqrt{3}}{2}$

178. $\frac{1}{2}$

179. $\frac{\sqrt{2}}{4}$

180. $\frac{1}{8}$

181. $\frac{63}{65}$

182. $\frac{9}{7}$

183. $$\cot(\theta+\phi) = \frac{1}{\tan(\theta+\phi)} = \frac{1-\tan\theta\tan\phi}{\tan\theta+\tan\phi} = \frac{1-\tan\theta\tan\phi}{\tan\theta+\tan\phi}\cdot\frac{\cot\theta\cot\phi}{\cot\theta\cot\phi} = \frac{\cot\theta\cot\phi-1}{\cot\theta+\cot\phi}$$

184. $$\cot 2\theta = \frac{1}{\tan 2\theta} = \frac{1-\tan^2\theta}{2\tan\theta} = \frac{1-\tan^2\theta}{2\tan\theta}\cdot\frac{\cot^2\theta}{\cot^2\theta} = \frac{\cot^2\theta-1}{2\cot\theta}$$

185. $$\frac{(\sin\theta+\cos\theta)^2-1}{\sin 2\theta} = \frac{\sin^2\theta+2\sin\theta\cos\theta+\cos^2\theta-1}{2\sin\theta\cos\theta} = \frac{2\sin\theta\cos\theta}{2\sin\theta\cos\theta} = 1$$

186. $$\frac{\sin 2\theta}{2\sin\theta} = \frac{2\sin\theta\cos\theta}{2\sin\theta} = \cos 2\left(\frac{\theta}{2}\right) = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}$$

187. $$\sin 3\theta = \sin(\theta+2\theta) = \sin\theta\cos 2\theta + \cos\theta\sin 2\theta = \sin\theta(1-2\sin^2\theta) + \cos\theta(2\sin\theta\cos\theta)$$
$$= \sin\theta - 2\sin^3\theta + 2\sin\theta\cos^2\theta = \sin\theta - 2\sin^3\theta + 2\sin\theta(1-\sin^2\theta)$$
$$= 3\sin\theta - 4\sin^3\theta$$

TRIGONOMETRIC EQUATIONS

188. $\theta = \frac{\pi}{4} + 2\pi n, \quad \theta = \frac{3}{4}\pi + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots$

189. $\theta = \frac{3}{4}\pi + \pi n, \quad n = 0, \pm 1, \pm 2, \ldots$

190. $\theta = \frac{\pi}{2} + \pi n, \quad n = 0, \pm 1, \pm 2, \ldots$

191. $\theta = \frac{4}{3}\pi + 2\pi n, \quad \theta = \frac{5}{3}\pi + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots$

192. $\theta = \frac{\pi}{3},\ \frac{2}{3}\pi,\ \frac{4}{3}\pi,\ \frac{5}{3}\pi$

193. $\theta = 0,\ \frac{\pi}{3}, \frac{2}{3}\pi,\ \pi,\ \frac{4}{3}\pi,\ \frac{5}{3}\pi$

194. no solution

195. $\theta = \frac{\pi}{3},\ \frac{5}{12}\pi,\ \frac{5}{6}\pi,\ \frac{11}{12}\pi,\ \frac{4}{3}\pi,\ \frac{17}{12}\pi,\ \frac{11}{6}\pi,\ \frac{23}{12}\pi$

196. $\theta = \frac{\pi}{6},\ \frac{5}{6}\pi,\ \frac{7}{6}\pi,\ \frac{11}{6}\pi$

197. $\theta = \frac{2}{3}\pi,\ \frac{4}{3}\pi, \quad \theta = 0$

198. $\theta = \frac{\pi}{6},\ \frac{5}{6}\pi,\ \frac{7}{6}\pi,\ \frac{11}{6}\pi$

199. $\theta = \frac{3}{2}\pi, \quad \theta = 0$

200. $\theta = 0,\ \frac{\pi}{2},\ \pi,\ \frac{3}{2}\pi, \quad \theta = \frac{\pi}{4},\ \frac{5}{4}\pi$

201. $\theta = \frac{3}{8}\pi,\ \frac{7}{8}\pi,\ \frac{11}{8}\pi,\ \frac{15}{8}\pi$

202. $\theta = \frac{2}{3}\pi,\ \frac{5}{6}\pi,\ \frac{5}{3}\pi,\ \frac{11}{6}\pi$

203. $\theta = \frac{\pi}{9},\ \frac{5}{9}\pi,\ \frac{7}{9}\pi,\ \frac{11}{9}\pi,\ \frac{13}{9}\pi,\ \frac{17}{9}\pi$

204. $\theta = 0,\ \pi$

205. $\theta = \frac{\pi}{6},\ \frac{5}{6}\pi$

206. $\theta = 0$

207. $\theta = 0$