MPE Review
Section III: Logarithmic & Exponential Functions

FUNCTIONS AND GRAPHS

To specify a function $y = f(x)$, one must give a collection of numbers **D**, called the **domain** of the function, and a **procedure** for determining exactly one number y from each number x in **D**. The procedure associated with a function can be given by a graph, by a table, by an expression or equation, or by a verbal description. When the domain of a function is not explicitly stated, it is understood to be the collection of all numbers to which the procedure for the function can be applied to obtain another number. The notation $f(a)$, where a is a number from the domain of f, denotes the number obtained by applying the procedure for f to the number a. When a is not in the domain of f, we say that $f(a)$ **is not defined**.

Example:

What is the domain of the function $g(t) = \sqrt{4 - t^2}$?

Solution:

The computational procedure for g can be applied to all numbers that do not require taking the square root of a negative number. Therefore, the domain of g is all numbers t such that $-2 \le t \le 2$.

Example:

Let $f(x) = \dfrac{x+3}{2x-1}$ and $a = 5$. Find $f(a)$ or state that a is not in the domain of f.

Solution:

The computational procedure given by the equation for f can be applied to the number $a = 5$ to obtain a real number. Therefore, 5 is in the domain of f and

$$f(5) = \frac{5+3}{2(5)-1} = \frac{8}{9}.$$

Problems:

1. What is the domain of the function $g(x) = \dfrac{\sqrt{9 - x^2}}{x - 2}$?

2. Let $h(x) = \dfrac{x^2 - 3x - 10}{x+1}$.

(a) Let $c = 4$. Find $h(c)$ or state that $h(c)$ is not defined.

(b) Find $h(-1)$ or state that -1 is not in the domain of h.

3. Let $g(x) = \dfrac{2x+1}{x-3}$. Find $g\left(-\dfrac{1}{2}\right) + g(2)$.

A graph in a Cartesian coordinate system specifies a function if and only if every line perpendicular to the *x*-axis intersects the graph in no more than one point. When a function is specified by a graph, its domain is the projection of the graph onto the *x*-axis and its range is the projection of the graph onto the *y*-axis.

To find $f(a)$ from the graph of f, locate a on the *x*-axis, find the point *P* where a vertical line through a intersects the graph, and, finally, locate the point $f(a)$ where the horizontal line through *P* intersects the *y*-axis.

When a number b is given, all of the numbers a such that $f(a) = b$ can be found from the graph of f. First, locate b on the vertical axis. Then, find all points *P* where a horizontal line through b intersects the graph. Finally, for each one of these points of intersection *P*, locate the point where the vertical line through *P* intersects the *x*-axis.

Example:

The graph of a function y = g(x) is shown in Figure F1.

(a) Find $g(2)$ or state that 2 is not in the domain of g.

(b) Find $g(-4)$ or state that -4 is not in the domain of g.

(c) Find all numbers a such that $g(a) = 1$ or state that 1 is not in the range of g.

(d) Find all numbers c such that $g(c) = 4$ or state that 4 is not in the range of g.

Solution:

(a) Since the vertical line through the point $x = 2$ on the *x*-axis does not intersect the graph, $x = 2$ is not in the domain of g.

(b) The vertical line through the point $x = -4$ on the *x*-axis intersects the graph at the point $P(-4, 1.5)$. The horizontal line through *P* intersects the *y*-axis at $y = 1.5$. Therefore, $g(-4) = 1.5$.

(c) The horizontal line through the point $y = 1$ on the *y*-axis intersects the graph at the points $Q(-2, 1)$ and $R(3.5, 1)$. The vertical lines through these points meet the *x*-axis at the points $x = -2$ and $x = 3.5$, respectively. Therefore, $g(-2) = 1$ and $g(3.5) = 1$.

(d) The horizontal line through the point $y = 4$ does not intersect the graph, so there is no number c for which $g(c) = 4$. The number 4 is not in the range of g.

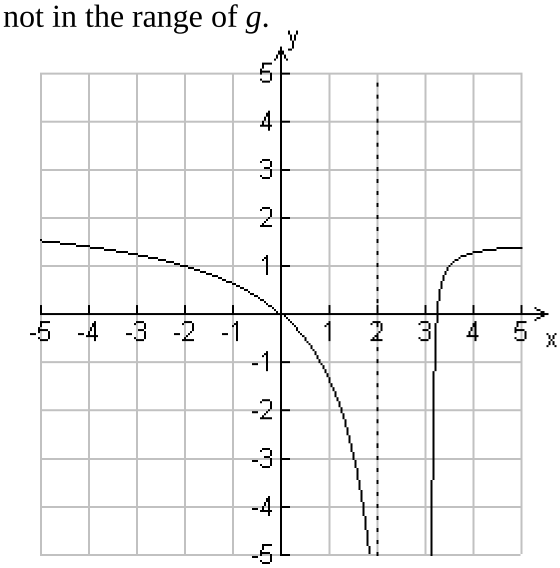


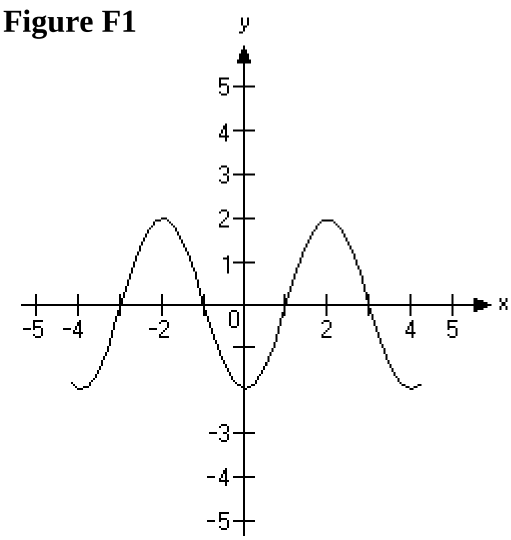
Figure F1

Problems:

4. The graph of a function $y = f(x)$ is shown in Figure F2.

(a) Find $f(-3)$ or state that -3 is not in the domain of f.

(b) Find $f(0)$ or state that 0 is not defined.



5. The graph of a function $y = g(x)$ is shown in Figure F3.

(a) Find all numbers a, such that $g(a) = 2$ or state that 2 is not in the range of g.

(b) Find all numbers c, such that $g(c) = 3$ or state that 3 is not in the range of g.

Figure F2

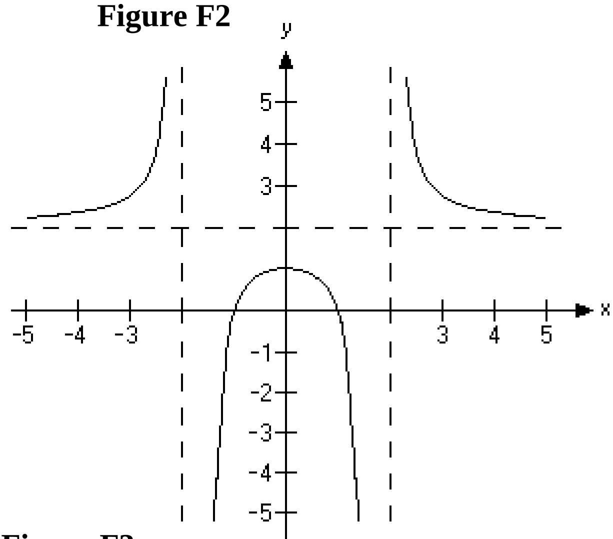


Figure F3

EXPONENTIAL FUNCTIONS

Functions given by the expressions of the form $f(x) = b^x$, where b is a fixed positive number and $b \neq 1$, are called **exponential functions**. The number b is called the **base** of the exponential function. Every exponential function of this form has all real numbers as its domain and all positive real numbers as its range. Exponential functions can be evaluated for integer values of x by inspection or by arithmetic calculation, but for most values of x, they are best evaluated with a scientific calculator.

The graphs of exponential functions $f(x) = b^x$ all have y-intercept 1, include the points $(1, b)$ and $\left(-1, \frac{1}{b}\right)$, and are increasing when $b < 1$. When $b > 1$, the negative x-axis is an asymptote of the graph of $f(x) = b^x$ and when $b < 1$, the positive x-axis is an asymptote. Use the graphing calculator to investigate the graphs of the exponential functions. Learn to recognize the graphs of exponential functions and to determine the base of an exponential function from its graph.

Problems:
Use a scientific calculator to evaluate the exponentials and logarithms in the following problems.

6. Find $10^{-0.345}$.

7. Find $e^{-\pi}$.

8. Let $f(x) = (5.4)^x$. Find $f\left(\frac{4}{5}\right)$.

9. The graph of an exponential function $y = b^x$ is shown in Figure F4.
What is the equation for this function?

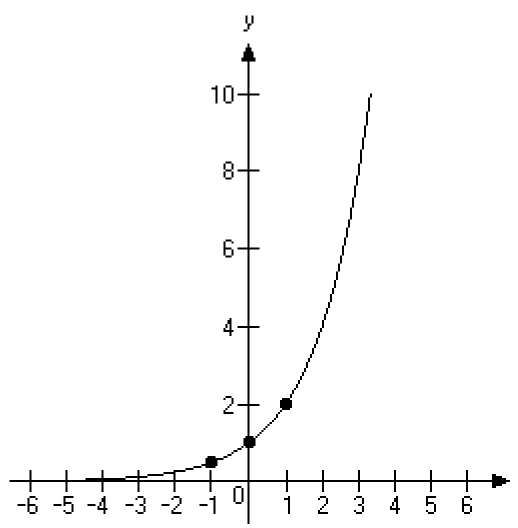


Figure F4

10. The graph of an exponential function $f(x) = b^x$ is shown in Figure F5.
What is the equation for this function?

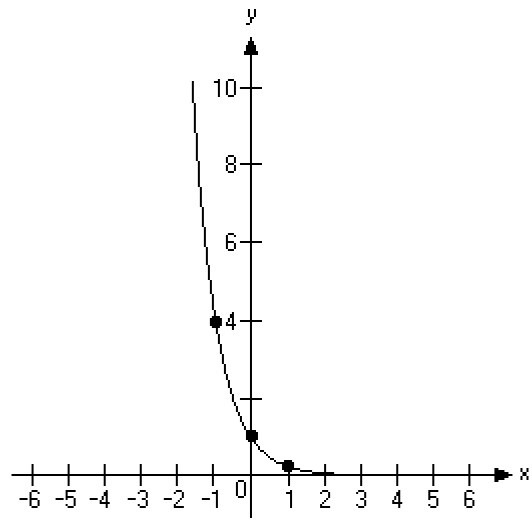


Figure F5

BASIC OPERATONS ON FUNCTIONS

We can add, subtract, multiply, and divide functions in much the same way as we do with real numbers. Let *f* and *g* be two functions. The **sum** $f + g$, the **difference** $f - g$, the **product** fg, and the **quotient** f/g are functions whose domains are the set of all real numbers common to the domains of *f* and *g*, and defined as follows:

Sum: $(f+g)(x) = f(x) + g(x)$
Difference: $(f-g)(x) = f(x) - g(x)$
Product: $(fg)(x) = f(x)\cdot g(x)$
Quotient: $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$, provided $g(x) \neq 0$

Example:

Let $f(x) = \dfrac{2x+3}{3x+2}$ and $g(x) = \dfrac{4x}{3x-2}$.

(a) Find $f + g$.
(b) Find $f - g$.
(c) Find fg.
(d) Find f/g.

Solution:

(a) $(f+g)(x) = f(x) + g(x) = \dfrac{2x+3}{3x+2} + \dfrac{4x}{3x-2} = \dfrac{(2x+3)(3x-2) + 4x(3x+2)}{(3x+2)(3x-2)} = \dfrac{18x^2 + 13x - 6}{9x^2 - 4}$.

(b) $(f-g)(x) = f(x) - g(x) = \dfrac{2x+3}{3x+2} - \dfrac{4x}{3x-2} = \dfrac{(2x+3)(3x-2) - 4x(3x+2)}{(3x+2)(3x-2)} = \dfrac{-6x^2 - 3x - 6}{9x^2 - 4}$.

(c) $(fg)(x) = f(x) \cdot g(x) = \frac{2x+3}{3x+2} \cdot \frac{4x}{3x-2} = \frac{8x^2+12x}{9x^2-4}$.

(d) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\frac{2x+3}{3x+2}}{\frac{4x}{3x-2}} = \frac{2x+3}{3x+2} \cdot \frac{3x-2}{4x} = \frac{6x^2+5x-6}{12x^2+8x}$.

Problems:

11. Let $r(x) = 7$ and $s(x) = \frac{3x-1}{x+3}$.

(a) Find $r + s$. **(b)** Find $r - s$. **(c)** Find $s - r$. **(d)** Find rs. **(e)** Find r/s. **(f)** Find s/r.

COMPOSITION AND INVERSE FUNCTIONS

The composition of two functions $y = f(x)$ and $y = g(x)$ is denoted $y = (f \circ g)(x)$ or $y = f(g(x))$. To evaluate $(f \circ g)(a)$, first evaluate $g(a)$ to get a number b and then evaluate $f(b)$ to get $f(b) = f(g(a)) = (f \circ g)(a)$. The domain of $f \circ g$ is all numbers a in the domain of g such that $b = g(a)$ is in the domain of f. To obtain an expression for $(f \circ g)(x)$ from expressions for f and g, replace the variable x in the expression for $f(x)$ with the expression for $g(x)$ and simplify the result. The domain of $f \circ g$ may or may not be all of the numbers that can be substituted into the simplified expression for the composite function $f \circ g$.

Example:

Let $g(x) = \sqrt{x-3}$ and $h(x) = x^2 + x - 3$.

(a) Find $(h \circ g)(4)$.

(b) Find $(g \circ h)(-2)$.

Solution:

(a) Since $g(4) = \sqrt{4-3} = \sqrt{1} = 1$, we have

$$(h \circ g)(4) = h(g(4)) = h(1) = (1)^2 + (1) - 3 = 1 + 1 - 3 = -1.$$

(b) Since $h(-2) = (-2)^2 + (-2) - 3 = -1$ and the number $b = -1$ is not in the domain of g, $(g \circ h)(-2)$ is not defined.

Example:

Let $F(x) = x + 3$ and $G(x) = x^2 - x + 1$.

(a) Find an expression for the composition $(F \circ G)(x)$.

(b) Find an expression for the composition $(G \circ F)(x)$.

Solution:

(a) $(F \circ G)(x) = F(G(x)) = G(x) + 3 = (x^2 - x + 1) + 3 = x^2 - x + 4$

(b) $(G \circ F)(x) = G(F(x)) = [F(x)]^2 - F(x) + 1 = (x+3)^2 - (x+3) + 1 = x^2 + 5x + 7$

Problems:

12. Let $F(x) = \sqrt{x-2}$ and $G(x) = x^2 - 11$.

(a) Find $(F \circ G)(2)$. **(b)** Find $(G \circ F)(6)$.

13. Let $f(x) = \frac{2}{3}x + 1$ and $g(x) = 9x^2 - 12x + 6$

(a) Find an expression for $(g \circ f)(x)$. **(b)** Find an expression for $(f \circ g)(x)$.

14. Let $G(x) = \dfrac{2x^5 + 3}{6x^5}$ and $H(x) = \sqrt[5]{\dfrac{6x}{2x+3}}$.

(a) Find an expression for $G(H(x))$. **(b)** Find an expression for $H(G(x))$.

Many, but not all, functions f are specified by a procedure that can be reversed to obtain a new function g. When this is possible, we say that ***f*** **has an inverse function** and that ***g*** **is the inverse function for** ***f***. A function f has an inverse function if and only if every horizontal line intersects the graph of f in no more than one point. When a function f has an inverse function g, the graph of g is the reflection of the graph of f through the line $y = x$.

Example:

The graph of a function F is shown in Figure F6. Sketch the graph of the inverse function G or state that F does not have an inverse function.

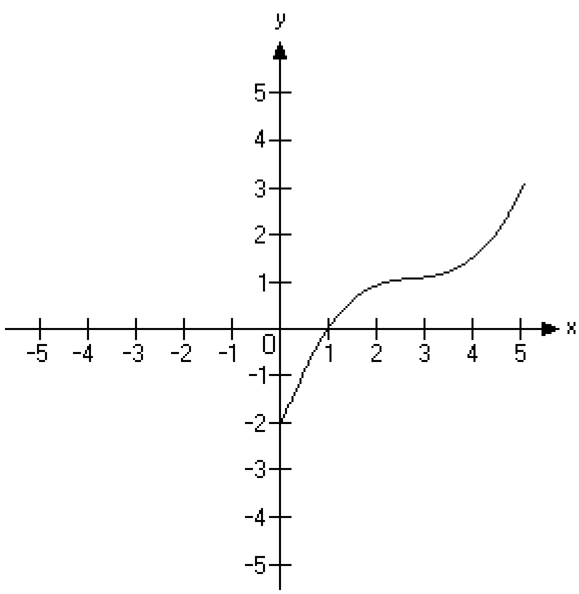


Figure F6

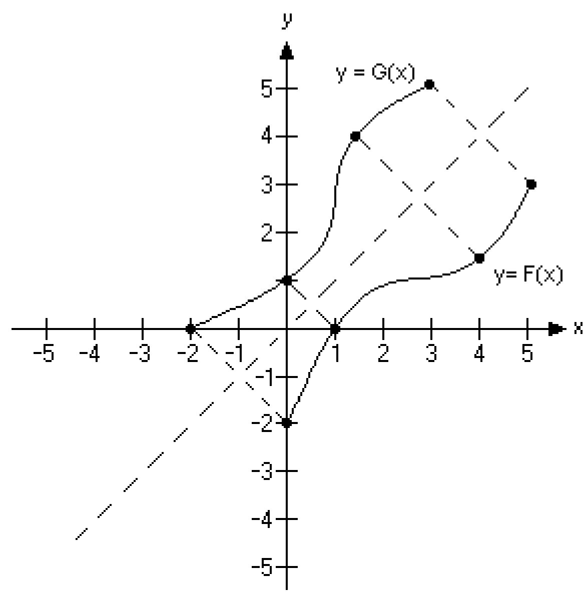


Figure F7

Solution:

The Horizontal Line Test shows that F (Figure F6) has an inverse function G. The graph of G (Figure F7) is the reflection of the graph of F through the diagonal line $y = x$. Plot a few points on the graph of G by reflecting points on the graph of F through the line $y = x$. Using these points as a guide, sketch the reflection of the graph of F through the line $y = x$ and obtain the graph of G as shown in Figure F7.

Example:

Each of the functions $y = x^3$ and $y = \sqrt[3]{x}$ is the inverse of the other. Construct the graphs of these two functions on the same coordinate system, either manually or by using a graphics calculator. Observe that the graph of each is the reflection of the graph of the other through the diagonal line $y = x$. If you use a graphing calculator, be sure to establish a coordinate system in which the units of distance on the two axes are the same.

Problems:

15. Sketch the graph of the function $f(x) = \sqrt{x}$. Determine whether f has an inverse function by examining its graph.

16. Sketch the graph of the function $F(x) = \dfrac{2x+3}{x-1}$. Determine whether F has an inverse function by examining its graph.

17. Sketch the graph of the function $G(x) = x^3 - x$. Determine whether G has an inverse function by examining its graph.

The procedure for the inverse function G of a function F is the reverse of the procedure for F. Intuitively, then, if we begin with a number x in the domain of F, apply the procedure for F to x and obtain a number w, and then apply the procedure for G (which is the reverse of the procedure for F) to this number w, we should be back at the number x again. Indeed, two functions f and g are a pair of inverse functions if and only if for every x in the domain of f, $(g \circ f)(x) = g(f(x)) = x$ and for every x in the domain of g, $(f \circ g)(x) = f(g(x)) = x$. This fact can be used to determine algebraically whether a given pair of functions are inverses.

Example:

Determine whether $f(x) = \dfrac{1}{x-3}$ and $g(x) = \dfrac{3x+1}{x}$ are inverse functions by computing their compositions.

Solution:

Since $f(g(x)) = \dfrac{1}{g(x)-3} = \dfrac{1}{\frac{3x+1}{x}-3} = \dfrac{x}{(3x+1)-3x} = x$

and $g(f(x)) = \dfrac{3f(x)+1}{f(x)} = \dfrac{3\left(\frac{1}{x-3}\right)+1}{\left(\frac{1}{x-3}\right)} = \dfrac{3+(x-3)}{1} = x$,

the functions f and g are a pair of inverse functions.

Example:

Determine whether $P(x) = \dfrac{x-2}{x-3}$ and $Q(x) = \dfrac{2x-3}{x-1}$ are inverse functions by computing their compositions.

Solution:

Since $P(Q(x)) = \dfrac{Q(x)-2}{Q(x)-3} = \dfrac{\left(\frac{2x-3}{x-1}\right)-2}{\left(\frac{2x-3}{x-1}\right)-3} = \dfrac{(2x-3)-2(x-1)}{(2x-3)-3(x-1)} = \dfrac{2x-3-2x+2}{2x-3-3x+3} = \dfrac{-1}{-x} = \dfrac{1}{x} \neq x$,

the functions P and Q are not a pair of inverse functions.

Problems:

Determine whether the following pairs of functions are inverses by computing their compositions.

18. $h(x) = \dfrac{-8}{x+4}$ and $k(x) = \dfrac{-4x-8}{x}$

19. $F(x) = \dfrac{-3x+5}{4x}$ and $G(x) = \dfrac{4x}{-3x+5}$

20. $g(x) = \dfrac{x}{x+1}$ and $p(x) = \dfrac{x}{x-1}$

21. $r(x) = \sqrt[5]{\dfrac{-8x}{11x+1}}$ and $s(x) = \dfrac{-x^5}{11x^5+8}$

When a function $y = f(x)$ has an inverse function $y = g(x)$, an equation for the inverse function can, in principle, be found by solving the equation $x = f(y)$. In practice, however, it may not be possible to solve this equation by familiar methods.

Example:

Find an equation for the inverse function for $F(x) = \dfrac{5x+4}{2x-7}$.

Solution:

Sketch the graph of *F*, preferably on a graphing calculator, and verify that *F* has an inverse function. Since *F* is given by the equation $y = \dfrac{5x+4}{2x-7}$, we can find an equation for its inverse function *G* by solving the equation

$$x = \frac{5y+4}{2y-7}$$

for *y*. We have

$$\begin{aligned} 2xy - 7x &= 5y + 4 \\ 2xy - 5y &= 7x + 4 \\ (2x-5)y &= 7x + 4 \\ y &= \frac{7x+4}{2x-5}. \end{aligned}$$

The inverse function for *F* is $G(x) = \dfrac{7x+4}{2x-5}$.

Problems:

Verify from their graphs that each of the following functions has an inverse function. Find an equation for each of the inverse functions.

22. $y = \dfrac{3x}{2x-9}$

23. $S(x) = \dfrac{2}{\sqrt[3]{1-x}}$

24. $y = \left(\dfrac{2\sqrt[5]{x}+1}{\sqrt[5]{x}}\right)^3$

LOGARITHMIC FUNCTIONS

Since horizontal lines intersect the graphs of exponential functions $f(x) = b^x$ ($b > 0$, $b \neq 1$) in no more than one point, every exponential function has an inverse function. The inverse of the exponential function with base b, $f(x) = b^x$, is called the **logarithmic function base *b*** and is denoted $\log_b x$. The logarithmic function base e is called the **natural logarithm function** and is denoted $\ln(x)$. The logarithmic function base 10 is called the **common logarithm function** and is denoted $\log(x)$. These logarithmic functions can be evaluated directly by using a scientific calculator.

Problems:

25. Let $f(x) = \ln(x)$. Find $f(2.349)$.

26. Let $f(x) = \log(x)$. Find $f\left(0.294^{2/5}\right)$.

27. Calculate $\dfrac{\ln 77.1 - 10^{1.94}}{\log 3}$.

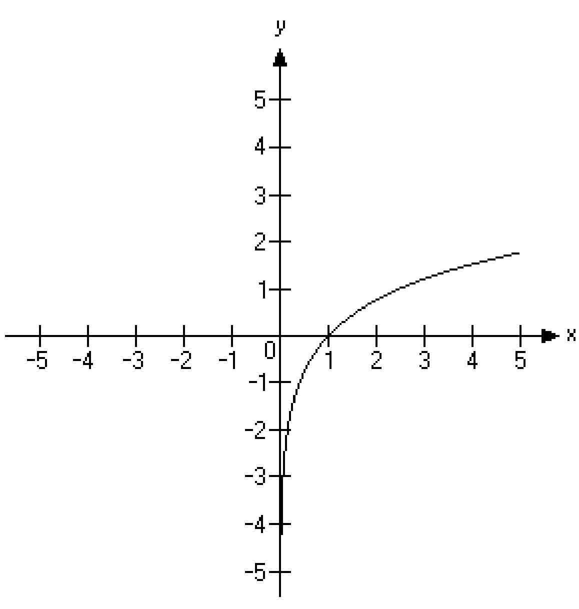
Since the graph of $y = \log_b x$ is the reflection of the graph of $y = b^x$ through the line $y = x$, properties of the graphs of the logarithmic functions can be inferred from properties of the graphs of the exponential functions. The graphs of logarithmic functions $f(x) = \log_b x$ all have x-intercept 1, include the points $(b, 1)$ and $\left(\frac{1}{b}, -1\right)$, and are increasing when $b > 1$ and decreasing when $b < 1$. When $b > 1$, the negative y-axis is an asymptote of the graph of $f(x) = \log_b x$ and when $b < 1$, the positive y-axis is an asymptote.

The graphs of the natural logarithm function and the common logarithm function can be generated easily on a graphing calculator. Some ingenuity is required to generate the graphs of other logarithmic functions on a graphing calculator. Learn to recognize the graphs of logarithmic functions and to determine the base of a logarithmic function from its graph.

Problems:

28. The graph of a logarithmic function is shown in Figure F8. Find the equation for this function.

29. The graph of a logarithmic function is shown in Figure F9. Find the equation for this function.



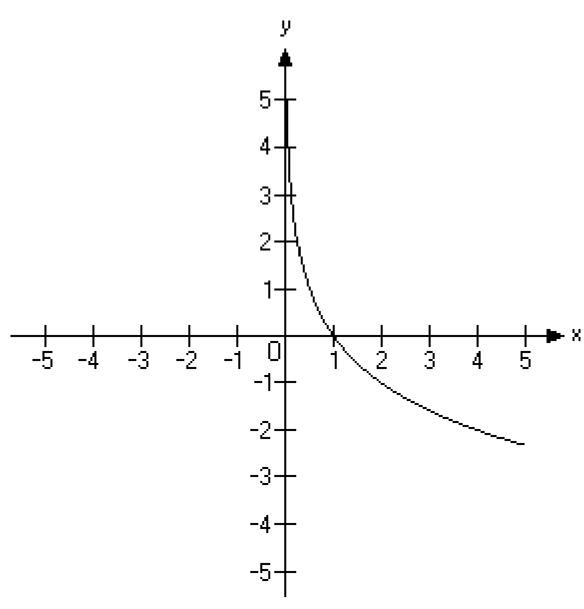


Figure F8 **Figure F9**

Since the exponential function $f(x) = b^x$ and the logarithmic function $g(x) = \log_b x$ are a pair of inverse functions, $f(a) = c$ if and only if $g(c) = a$. Therefore, the equations $b^a = c$ and $\log_b c = a$ ($b > 0$, $b \neq 1$) express the same relationship among the numbers a, b and c.

Example:

Rewrite the exponential equation $7^w = z$ as a logarithmic equation.

Solution:

The equivalent logarithmic equation is $w = \log_7 z$.

Example:

Rewrite the logarithmic equation $\log_{(1+x^2)} w = 3y$ as an exponential equation.

Solution:

The equivalent exponential equation is $w = (1 + x^2)^{3y}$.

Problems:

30. Rewrite $s^{4.2} = 11.3$ as a logarithmic equation.

31. Rewrite $\log_b 12 = a$ as an exponential equation.

32. Rewrite $a^{(1 + kt)} = S + R$ as a logarithmic equation.

33. Rewrite $\log_p (4 + x^2) = y$ as an exponential equation.

Some problems involving logarithmic functions can be readily solved by rewriting them in exponential form.

Example:

Solve the equation $\log_{7.23} x = 0.93$.

Solution:

Rewrite the logarithmic equation as the exponential equation $(7.23)^{0.93} = x$. Use a scientific calculator to find $x = 6.295$ (rounded to three decimal places).

Example:

Solve the equation $\log_x 81 = 4$.

Solution:

Rewrite the logarithmic equation as the exponential equation $x^4 = 81$. Apply the fourth root function to both sides of the equation to find $x = 81^{(1/4)} = 3$.

Example:

Solve the equation $\log_9 27 = x$.

Solution:

Rewrite the logarithmic equation as the exponential equation $9^x = 27$. Rewrite this exponential equation as $3^{2x} = 3^3$, so each member of the equation is a power of the same base. Equate the exponents and solve for x to find $x = {}^3/_2$.

Problems:

Solve each of the following equations.

34. $\log_{11} x = \dfrac{2}{3}$

35. $\log_x 11 = \dfrac{2}{3}$

36. $\log_8 128 = x$

The logarithmic functions satisfy the identities

$$\log_b AB = \log_b A + \log_b B \quad \text{(the Product Identity),}$$

$$\log_b\left(\frac{A}{B}\right) = \log_b A - \log_b B \quad \text{(the Quotient Identity) and}$$

$$\log_b A^p = p \log_b A \quad \text{(the Power Identity).}$$

where A and B may be positive numbers or variables, algebraic expressions or functions that take on positive values. These identities are used to write expressions involving logarithmic functions in different forms to suit different purposes.

Example:

Write the expression $\log_5 3(x + 4) - 4 \log_5 (x - 2) + 2 \log_5 (x^2 - 4)$ as a single logarithm and simplify.

Solution:

Use the identities for logarithmic functions to rewrite this expression:

$$\begin{aligned}\log_5 3(x+4) - 4\log_5(x-2) + 2\log_5\left(x^2-4\right) &= \left[\log_5 3(x+4) - \log_5(x-2)^4\right] + \log_5\left(x^2-4\right)^2 \\ &= \log_5 \frac{3(x+4)}{(x-2)^4} + \log_5\left(x^2-4\right)^2 \\ &= \log_5 \frac{3(x+4)\left(x^2-4\right)^2}{(x-2)^4}.\end{aligned}$$

Now simplify by factoring and dividing out common factors:

$$\begin{aligned}\log_5 3(x+4) - 4\log_5(x-2) + 2\log_5\left(x^2-4\right) &= \log_5 \frac{3(x+4)\left(x^2-4\right)^2}{(x-2)^4} \\ &= \log_5 \frac{3(x+4)(x+2)^2(x-2)^2}{(x-2)^4} \\ &= \log_5 \frac{3(x+4)(x+2)^2}{(x-2)^2}.\end{aligned}$$

Problems:

In the following problems, write the expression as a single logarithm and simplify.

37. $\dfrac{3}{5}\log_2 u + \dfrac{1}{4}\log_2 v - 3\log_2(2w)$

38. $3 \log_7 5(2s + 1) - 5 \log_7 s - 2 \log_7 (2s^2 - s - 1)$

39. $3\log(x^2 + 8xy + 12y^2) - 4\log(x + 6y) - \dfrac{2}{3}\log(x + 2y)$

In the following problems, write the expression as a sum of multiples of logarithms.

40. $\log_8 \dfrac{64\sqrt[3]{u^5 v^2}}{w^{-2}}$

41. $\ln \sqrt[3]{\dfrac{x^{1/3}\left(2x^2 + x + 5\right)}{(x+5)^2}}$

Since the exponential function $f(x) = b^x$ and the logarithmic function $g(x) = \log_b x$ of the same base b ($b > 0$, $b \neq 1$) are inverse functions, exponential and logarithmic functions satisfy the **Composition Identities**. For every positive number x,

$$f(g(x)) = b^{(\log_b x)} = x$$

and, for every number x,

$$g(f(x)) = \log_b (b^x) = x.$$

The Composition Identities also hold when x is replaced by any variable, algebraic expression or function that takes on values for which the compositions are defined. These identities are used to simplify expressions involving logarithms and exponential functions of the same base.

Example:

Simplify $7^{\left[\log_7(s-r)+\log_7(s+r)\right]}$.

Solution:

Using the identities for logarithmic, write the exponent as a single logarithm. The expression becomes

$$7^{\left[\log_7(s-r)+\log_7(s+r)\right]} = 7^{\log_7(s-r)(s+r)}$$

Use a Composition Identity to simplify further and get

$$7^{\left[\log_7(s-r)+\log_7(s+r)\right]} = 7^{\log_7(s-r)(s+r)} = (s-r)(s+r)\,.$$

Example:

Simplify $(a+b)\log_2 \sqrt{\dfrac{2^{(a+b)^2}}{2^{4ab}}}$.

Solution:

$$\sqrt{\frac{2^{(a+b)^2}}{2^{4ab}}} = \left[2^{(a+b)^2-4ab}\right]^{\frac{1}{2}} = \left[2^{a^2-2ab+b^2}\right]^{\frac{1}{2}} = \left[2^{(a-b)^2}\right]^{\frac{1}{2}} = 2^{\frac{1}{2}(a-b)^2}$$

By a Composition Identity,

$$(a+b)\log_2\left[2^{\frac{1}{2}(a-b)^2}\right] = (a+b)\frac{1}{2}(a-b)^2$$

The expression simplifies to $\dfrac{1}{2}(a+b)(a-b)^2$.

Problems:
Simplify the following expressions.

42. $(t+3)5^{\left[\log_5(t^2-5t+6)-\log_5(t^2-9)\right]}$

43. $(t-u)10^{\left[\log\left(t^2-u^2\right)+\frac{1}{3}\log\left(t^2+2ut+u^2\right)-\frac{5}{3}\log(t-u)\right]}$

44. $(a+b)^{\frac{1}{2}}\log_7\left[7^{(2a-3b)}\cdot 7^{(4b-a)}\right]$

45. $\log\left[10^{\left(5st+t^2\right)}\cdot 10^{\left(s^2-3st\right)}\right]+4^{\log_4 5st}$

SOLVING EQUATIONS

Many equations involving logarithmic functions can be solved by rewriting the equation so it has one logarithmic term on the left and a constant on the right, rewriting this equation as an equivalent exponential equation, and, finally, solving the exponential equation. Since this procedure can introduce extraneous solutions, all solutions to the exponential equation must be checked by substitution to determine whether they satisfy the original equation.

Example:

Solve $\log_2(x - 2) + \log_2(x + 1) = 2$.

Solution:

Use the Product Identity for Logarithms to write the equation as

$$\log_2(x - 2)(x + 1) = 2.$$

Rewrite this equation as the exponential equation

$$(x - 2)(x + 1) = 2^2,$$

or

$$x^2 - x - 6 = 0.$$

The solutions to this quadratic equation are $x = 3$ and $x = -2$. On checking by substitution in the original equation, we find that $x = 3$ is a solution and $x = -2$ is not a solution.

Example:

Solve $(\ln x)^2 + \ln x^2 - 8 = 0$.

Solution:

Use the Power Identity to rewrite this equation as

$$(\ln x)^2 + 2\ln x - 8 = 0.$$

Write this equation in factored form as

$$(\ln x + 4)(\ln x - 2) = 0.$$

In order for this product to be zero, we must have either

$$\ln x = -4 \text{ or } \ln x = 2.$$

Rewrite these two equations in exponential form and find that we must have either

$$x = e^{-4} = 0.01831564,$$

or

$$x = e^{2} = 7.38905610.$$

Check by substituting into the original equation and verify that both of these numbers are solutions.

Problems:
Solve the following equations.

46. $\log(x - 4) + \log(x + 2) = 0.84510$

47. $\log_{81}(5x - 1) - \log_{81}(x + 2) = \frac{1}{2}$

48. $\log(\log_2(3x + 7)) = 0.47713$

49. $2(\log_4 x)^2 + \log_4 x^3 = 5$

Many equations involving exponentials can be solved by rewriting the equation so it has one exponential term on the left and either a constant or another exponential term on the right, applying a logarithmic function to both sides of the equation, and, finally, simplifying and solving the resulting equation.

Example:
Solve $5^{(3x-2)} - 3 \cdot 2^{(x+1)} = 0$. Round off the answer to four decimal places.

Solution:
Rewrite the equation as

$$5^{(3x-2)} = 3 \cdot 2^{(x+1)}.$$

Apply the function ln or log (any logarithm function would do, but these are on the calculator) to both sides of the equation to find

$$\ln\left[5^{(3x-2)}\right] = \ln\left[3 \cdot 2^{(x+1)}\right].$$

Simplify this equation as

$$(3x - 2)\ln 5 = \ln 3 + (x + 1)\ln 2.$$

Now, solve this equation for x to find

$$(3\ln 5 - \ln 2)x = \ln 3 + \ln 2 + 2\ln 5,$$

and, rounded to four decimal places,

$$x = \frac{\ln 3 + \ln 2 + 2\ln 5}{3\ln 5 - \ln 2} = 1.2117$$

Example:
Solve $5^x - 4\left(5^{-x}\right) = 3$. Round off the answer to four decimal places.

Solution:
Multiply each term of the equation by 5^x and rewrite the equation as

$$5^{2x} - 4 = 3\left(5^x\right) \text{ or } \left(5^x\right)^2 - 3\left(5^x\right) - 4 = 0.$$

Write this equation in factored form as

$$\left(5^x + 1\right)\left(5^x - 4\right) = 0.$$

In order for this product to be zero, we must have either

$$5^x = -1 \text{ or } 5^x = 4.$$

From the equation $5^x = -1$ we find no solution, since for every real number x, $5^x > 0$. Apply a logarithmic functions to both members of the equation $5^x = 4$ and find $x \log 5 = \log 4$. Thus, rounded to four decimal places

$$x = \frac{\log 4}{\log 5} = 0.8614.$$

Problems:
Solve the following equations.

50. $3^{(2x + 1)} = 108$

51. $7 \cdot 3^{(8x - 1)} = 5^{-x}$

52. $2e^{x} - 5e^{-x} = 9$

53. $e^{(4x + 3)} = 5^{(x - 7)}$

54. $\frac{5^{(x+1)}}{7^{(x-3)}} = 3$

MATHEMATICAL MODELS

A mathematical equation which relates the variables involved in a phenomenon or situation is called a **mathematical model** for the situation. Questions about a phenomenon or situation can often be answered from a mathematical model.

Example:

The weight in pounds of a certain snowball rolling down a hill is given by

$$W = 3e^{0.006S},$$

where S is the distance (in feet) the ball has rolled. How much does the ball weigh initially (when $S = 0$)? How much does the ball weigh after it has rolled 1000 feet?

Solution:

When $S = 0$, $W = 3e^{0} = 3$ pounds.
When $S = 1000$, $W = 3e^{6} = 1210$ pounds (rounded).

Problems:

55. Nitrogen pentoxide is a solid that decomposes into the gases nitrogen dioxide and oxygen. The function

$$N(t) = 17e^{(-0.0005)t},$$

where t is the time in seconds and N is the number of grams of nitrogen pentoxide remaining in the sample at time t, is a mathematical model for the decomposition of a sample of 17 grams of nitrogen pentoxide.

(a) According to this model, how much nitrogen pentoxide remains after 1000 seconds?

(b) How long does this model predict it will take for 15 grams of nitrogen pentoxide to decompose from the sample (so 2 grams remain)?

56. The human ear is sensitive to sounds over a wide range of intensities. Sounds of intensity 10^{-12} watts per square meter are at the threshold of hearing. Sounds of intensity 1 watt per meter cause pain for most people. Because of this wide range of intensities and because the sensation of loudness seems to increase as a logarithm of intensity, the loudness D (in decibles) of a sound of intensity I is defined by the equation

$$D = 120 + 10 \log I.$$

(a) What is the loudness of a sound intensity $I = 10^{-12}$ watts per square meter, which is the threshold of human hearing?

(b) Constant exposure to sound of 90 decibles (or greater) endangers hearing. What is the intensity of sounds of loudness $D = 90$ decibles?

(c) A large rocket engine generates sounds of loudness 180 decibles. What is the intensity of the sound generated by such an engine?

(d) How loud (in decibles) is the sound of a rock band that produces sounds of intensity 0.83 watts per square meter?

57. The concentration C of medication in the bloodstream of a patient t hours after 50 milligrams of the medication is administered orally is given by the equation

$$C = 66\left(e^{-0.5t} - e^{-2t}\right).$$

What is the concentration of the medication in the patient's bloodstream 4 hours after the medication is administered?

III. LOGARITHMIC AND EXPONENTIAL FUNCTIONS
ANSWER SECTION

FUNCTIONS AND GRAPHS

1. $\{x \mid -3 \le x < 2 \text{ or } 2 < x \le 3\}$

2. **(a)** $h(4) = -\dfrac{6}{5}$

(b) - 1 is not in the domain of h.

3. – 5

4. **(a)** $f(-3) = 0$

(b) $f(0) = -2$

5. **(a)** 2 is not in the range of g.

(b) -3 and 3

EXPONENTIAL FUNCTIONS

6. 0.45186 (rounded)

7. 0.04321 (rounded)

8. 3.85403 (rounded)

9. $y = 2^x$

10. $f(x) = \left(\dfrac{1}{4}\right)^x$

BASIC OPERATONS ON FUNCTIONS

11. **(a)** $(r+s)(x) = \dfrac{10x+20}{x+3}$

(b) $(r-s)(x) = \dfrac{4x+22}{x+3}$

(c) $(s-r)(x) = \dfrac{-4x-22}{x+3}$

(d) $(rs)(x) = \dfrac{21x-7}{x+3}$

(e) $\left(\dfrac{r}{s}\right)(x) = \dfrac{7x+21}{3x-1}$

(f) $\left(\dfrac{s}{r}\right)(x) = \dfrac{3x-1}{7x+21}$

COMPOSITIONS AND INVERSE FUNCTIONS

12. **(a)** undefined
(b) -7

13. **(a)** $(g \circ f)(x) = 4x^2 + 4x + 3$
(b) $(f \circ g)(x) = 6x^2 - 8x + 5$

14. **(a)** $G(H(x)) = \dfrac{2x+1}{4x}$
(b) $H(G(x)) = \sqrt[5]{\dfrac{6x^5+9}{11x^5+3}}$

15. f has an inverse function.

16. F has an inverse function.

17. G does not have an inverse function.

18. are inverses

19. are not inverses

20. are not inverses

21. are inverses

22. $y = \dfrac{9x}{2x-3}$

23. $T(x) = \dfrac{x^3 - 8}{x^3}$

24. $y = \left(\dfrac{1}{\sqrt[3]{x} - 2}\right)^5$

LOGARITHMIC FUNCTIONS

25. 0.85399 (rounded)

26. -0.21266 (rounded)

27. -173.43863 (rounded)

28. $y = \log_{5/2} x$

29. $y = \log_{1/2} x$

30. $4.2 = \log_s 11.3$

31. $b^a = 12$

32. $\log_a (S + R) = 1 + kt$

33. $p^y = 4 + x^2$

34. $x = 4.94609$ (rounded)

35. $x = 36.48287$ (rounded)

36. $x = \dfrac{7}{3}$

37. $\log_2\left(\dfrac{u^{3/5}v^{1/4}}{8w^3}\right)$

38. $\log_7 \dfrac{125(2s+1)}{s^5(s-1)^2}$

39. $\log \dfrac{(x+2y)^{7/3}}{x+6y}$

40. $\dfrac{5}{3}\log_8 u + \dfrac{2}{3}\log_8 v + 2\log_8 w + 2$

41. $\dfrac{1}{3}\ln\left(2x^2 + x + 5\right) - \dfrac{2}{3}\ln(x+5) + \dfrac{1}{9}\ln x$

42. $t - 2$

43. $(t-u)^{1/3}(t+u)^{5/3}$

44. $(a+b)^{3/2}$

45. $t^2 + 7st + s^2$

SOLVING EQUATIONS

46. $x = 5$

47. no solution

48. $x = \dfrac{1}{3}$

49. $x = 4^{-5/2} = 0.03125$ or $x = 4$

50. $x = 1.63093$ (rounded)

51. $x = -0.081484$ (rounded)

52. $x = 1.60944$ (rounded)

53. $x = -5.96766$ (rounded)

54. $x = 18.8680$ (rounded)

MATHEMATICAL MODELS

55. **(a)** 10.3 grams
(b) 4,280 seconds

56. **(a)** 0 decibels
(b) 10^{-3} watts/square meter
(c) 10^{6} watts/square meter
(d) 119.2 decibels

57. $C = 8.9$ mg/ml