CSU Math Day 2005
Team Competition
11:20

1. What percent of the odd integers between 0 and 100 are multiples of 7?

   ANS: $\boxed{14}\ \% = \frac{7}{50}$ $(7, 21, 35, 49, 63, 77, 91)$

2. Given the circle $x^2 + x + y^2 - y = \frac{1}{2}$ in the xy plane, what are the equations of the two vertical tangent lines to this circle?

   ANS: $\boxed{x = -3/2}$ and $\boxed{x = 1/2}$ $\left(x + \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = 1$

3. On a number line, point B has coordinate 4.322 and point M has coordinate 2.718. Point M is the midpoint of segment AB. Find the coordinate of A.

   ANS: $\frac{4.322+x}{2} = 2.718$, Solution is: $x = \boxed{1.114}$

4. A baseball manager has selected 9 starters for a game. If the pitcher must bat last, the second baseman leads off, and the first baseman bats in the cleanup spot, how many different batting line-ups are possible?

   ANS: $6! = \boxed{720}$

5. A bag of chicken feed will feed 16 chickens for 30 days. For how many days will it feed 24 chickens?

   ANS: $\boxed{20}$ days $= \dfrac{16 \cdot 30}{24}$

6. What is the volume of a cube with surface area 216?

   ANS: $\boxed{216}$ $(216 = 6a^2$, Solution is: 6 and $6^3 = 216)$

7. Find all positive real roots of the equation $x^4 - 13x^2 + 36 = 0$.

   ANS: $x^4 - 13x^2 + 36 = 0$, Solution is: $-2, -3, \boxed{2}, \boxed{3}$

8. Where in quadrant 4 does the circle of radius 10 centered at the origin intersect the line passing through the origin with slope $-4/3$?

   ANS: $\left\{ \begin{array}{c} x^2 + y^2 = 100 \\ y = -\frac{4}{3}x \end{array} \right\}$, Solution is: $\boxed{(6, -8)}$, $(-6, 8)$

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9. In how many ways can the US Senate pick a committee of 3 from among its 100 members?

   ANS: $\binom{100}{3} = \boxed{161\,700}$

10. In 1945, Eilenberg and MacLane introduced which of the following terms: A) group, B) filum, or C) category?

    ANS: $\boxed{\text{C) category}}$

11. Two cars start 2 miles apart and drive toward each other. One car goes 40 mph, the other 50 mph. After how many seconds do the two cars meet?

    ANS: $\frac{2\,\text{mi}}{90\,\text{mi}/\,\text{h}} \cdot 60\,\text{min}/\,\text{h} \cdot 60\,\text{s}/\,\text{min} = \boxed{80}$ seconds

12. Find all the integer solutions of the equation $x^4 - x^2 - 12 = 0$.

    ANS: $x = \boxed{2, -2}$ since $x^4 - x^2 - 12 = (x-2)(x+2)(x^2+3)$

13. What is the prime factorization of 1234?

    ANS: $1234 = \boxed{2 \times 617}$

14. What is the surface area of a sphere of volume 288π?

    ANS: $288\pi = \frac{4\pi r^3}{3}$, Solution is: $6, -3i\sqrt{3} - 3, 3i\sqrt{3} - 3$ so $4\pi 6^2 = \boxed{144\pi}$

15. A triangle has sides of length 6, 5, and 5. Find the area of the triangle.

    ANS: Altitude is 4, so $Area = \frac{1}{2}(6)(4) = \boxed{12}$

Tiebreaker Question

Which of the following three numbers is largest: $a = \frac{23}{46}$, $b = \frac{85}{153}$, or $c = \frac{52}{117}$?

ANS: $a = \frac{23}{46} = \frac{1}{2}$, $\boxed{b = \frac{85}{153} = \frac{5}{9}}$, $c = \frac{52}{117} = \frac{4}{9}$

CSU Math Day 2005
Team Competition
11:40

1. A 7-symbol license plate has 4 letters and 3 digits. What is the probability that at least one of the digits is a 5?

   ANS: $1 - \left(\frac{9}{10}\right)^3 = \boxed{\frac{271}{1000}}$

2. A punch bowl at a Halloween party is the shape of a truncated cone of height 12 inches, base diameter 9 inches, and top diameter 15 inches. The orange plastic punch cups are also truncated cones of height 4 inches, base diameter 3 inches, and top diameter 5 inches. How many cups of punch will the punch bowl hold?

   ANS: $3^3 = \boxed{27}$

3. List the following three numbers in increasing order: $\sqrt{3}$, $\sqrt[3]{5}$, $\sqrt[6]{26}$.

   ANS: $\boxed{\sqrt[3]{5}} = 1.7100 < \boxed{\sqrt[6]{26}} = 1.7212 < \boxed{\sqrt{3}} = 1.7321$

   Note: $\left(\sqrt[3]{5}\right)^6 = 25 < \left(\sqrt[6]{26}\right)^6 = 26 < \left(\sqrt{3}\right)^6 = 27$

4. What did the Norwegian mathematician Niels Henrik Abel prove about general fifth-degree polynomials?

   ANS: $\boxed{\text{Cannot be solved}}$ in terms of radicals involving the coefficients

5. If eggs weigh 3 oz each and a dozen eggs cost 90¢, what is the cost of a pound of eggs? (16oz = 1 pound)

   ANS: $\boxed{40}$ ¢ $= \frac{90c}{12e} \times \frac{1e}{3\,\text{ozf}} \times \frac{16\,\text{ozf}}{1\,\text{lbf}} = 40\frac{c}{\text{lbf}}$

6. What is the sum of the roots of the polynomial $x^2 - x - 20$?

   ANS: $\boxed{1}$ since $(x-a)(x-b) = x^2 - x\,(a+b) + ab \Rightarrow a+b = 1$

7. Let $S = \{1, 2, 3, 4, 5, 6\}$ be the set of the first 6 natural numbers. How many of the 64 subsets of S contain the numbers 2 and 5?

   ANS: $\boxed{16}$ or $\boxed{\text{One-fourth}}$

8. How far apart are the two points at which the curves $x^2 + y^2 = 169$ and $x = 12$ intersect?

   ANS: $\left\{ \begin{array}{l} x = 12 \\ x^2 + y^2 = 169 \end{array} \right\}$, Points $(12, 5)$ and $(12, -5)$, distance $\boxed{10}$

Team Competition
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9. Eighteen liters of paint are required to paint the outside of a cubic box of volume $64\,\text{m}^3$. How much paint is needed to paint the outside of a cubic box of volume $125\,\text{m}^3$?

   ANS: $\boxed{25}$ liters $\frac{25}{16} \cdot 16 = 25$

10. What is the area of the largest triangle with perimeter 12?

    ANS: $\boxed{4\sqrt{3}}$ = area of equilateral triangle of edge 4

11. From a group of six students living in the same neighborhood, a social committee is to be appointed. Given that the committee must have at least four members, how many different committees can be formed?

    ANS: $\binom{6}{4} + \binom{6}{5} + \binom{6}{6} = \boxed{22}$

12. How many 4-person committees can be made out of a group of 8 people?

    ANS: $\displaystyle\binom{8}{4} = \boxed{70}$

13. Find x so that the average of the four numbers 23, 14, 43, and x is 30.

    ANS: $\frac{23+14+43+x}{4} = 30$, Solution is: $x = \boxed{40}$

14. What is the next term in the sequence that begins $8, 15, 23, 38, 61$?

    ANS: $\boxed{99}$ (each new term is the sum of the previous two)

15. At the local Dairy Queen, the "Monster Sundae" can be ordered with any one of 6 flavors of ice cream plus any combination of 6 toppings. If you order one such sundae every Saturday, how many weeks will it be before you must order the same sundae twice?

    ANS: $6 \times 2^6 = \boxed{384}$

Tiebreaker Question

A rectangle has perimeter 36 and area 80. What are the dimensions of the rectangle?
ANS: $\boxed{\text{8 by 10}}$

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1. Art spent two-thirds his money plus a dollar, then two-thirds the remainder plus a dollar, then two-thirds of what he had left plus a dollar, when he found that he had \$2.00 remaining. How much did he start with?

   ANS: $(((2+1)\,3+1)\,3+1)\,3 = \$\boxed{93}$

2. What is the prime factorization of 9797?

   ANS: $\boxed{97 \cdot 101} = 9797$

3. Norbert Wiener was born in Columbia, Missouri in 1894. He had an extraordinarily wide range of interests and contributed to many areas of mathematics, but is best known for his work in A) Cybernetics, B) Magnetics, or C) Dietetics.

   ANS: $\boxed{\text{A) Cybernetics}}$

4. If x is a positive angle measured in radians, then which is greater, x or $\sin(x)$?
   ANS: x

5. An octahedron is a regular solid whose 8 faces are equilateral triangles. What is the distance between a pair of opposite vertices, assuming each edge of the octahedron is of length 1?

   ANS: $\boxed{\sqrt{2}}$

6. A gold watch has been reduced by 10%, then by 20%, and finally sold for \$864.00. What was the original price?

   ANS: $\$\boxed{1200}$ since $1200 \times 0.9 \times 0.8 = 864$

7. A spherical balloon's diameter increases by 50%. By what percentage does the surface area change?

   ANS: $\boxed{125}$ % since $\frac{4\pi\left(\frac{3}{2}r\right)^2}{4\pi r^2} = \frac{9}{4}$ so increase is $\frac{5}{4}$ or 125%

8. How many different 6-place license plates are possible if the first three places are letters and the last 3 places are digits?

   ANS: $26^3 \times 10^3 = \boxed{17\,576\,000}$

Team Competition
12:00

9. A quadrilateral kite is made with a right angle at the top, angles of 2α left and right, and an angle of α at the bottom. What is α?

   ANS: $5\alpha = 360° - 90° \Longrightarrow \alpha = \boxed{54°}$ or $\boxed{\frac{3\pi}{10}}$

10. Allison scored 78 on the first exam and 74 on the second exam. What must she average on the next two exams to bring her average for the four exams up to 85?

    ANS: $85 = \frac{78+74+2x}{4}$, Solution is: $\boxed{94}$

11. How many distinct complex roots does the polynomial $x^6 + 1$ have?

    ANS: $\boxed{6}$

12. Ann keeps flies and spiders in a box in her dorm room during the Halloween season. There are a total of 14 creatures with 102 legs. How many flies and how many spiders does she have? (Flies have 6 legs and spiders have 8)

    ANS: $5 \cdot 6 + 9 \cdot 8 = 102$ so $\boxed{5}$ flies and $\boxed{9}$ spiders

13. What is the diameter of a circle with area $8\pi\,\text{cm}^2$?

    ANS: $8\pi = \pi r^2$, Solution is: $r = 2\sqrt{2}$, so diameter is $\boxed{4\sqrt{2}}$

14. A piece of string 49 inches long is cut into two pieces so that one piece is 5 inches shorter than the other. What are the lengths of the two pieces?

    ANS: $x + (x + 5) = 49$, Solution is : $x = \boxed{22}$ in, $x + 6 = \boxed{27}$ in

15. How many positive three-digit integers less than 400 are divisible by 7?

    ANS: $15 \times 7 = 105, \ldots, 57 \times 7 = 399$, $57 - 15 + 1 = \boxed{43}$

Tiebreaker Question

List the following three numbers in increasing magnitude: $4\sqrt{2}$, $3\sqrt{3}$, $2\sqrt{7}$?
ANS: $\boxed{3\sqrt{3}} = 5.196\,2\;4 < \boxed{2\sqrt{7}} = 5.291\,5 < \boxed{4\sqrt{2}} = 5.656\,9$ (Note that $\left(3\sqrt{3}\right)^2 = 27 < \left(2\sqrt{7}\right)^2 = 28 < \left(4\sqrt{2}\right)^2 = 32$.)

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1. The White Rabbit says that the Cat is lying. The Cat says that Alice is lying. Alice says that the White Rabbit and the Cat are both lying. Who is telling the truth?
   ANS: The $\boxed{\text{Cat}}$

2. I travel 2 miles north, 8 miles west, and 8 miles south. Assuming a flat earth, how far from my starting point am I?
   ANS: $\boxed{10}$ miles

3. How many committees of 2 men and 2 women can be formed from a group of 6 men and 5 women?
   ANS: $\binom{6}{2} \times \binom{5}{2} = \boxed{150}$

4. What is the smallest number greater than 50 has the property that when divided by 18 the remainder is 1 and when divided by 20 the remainder is also 1?
   ANS: $\boxed{181} = 18 \times 10 + 1 = 20 \times 9 + 1$

5. A shirt was marked up 30%, then marked down 30% for a sale price of \$36.40. What was the original price?
   ANS: \$ $\boxed{40}$ $\times 1.3 \cdot .7 = 36.4$

6. Which of the following best describes how many 7-symbol license plates are possible if the symbols come from the letters A-Z together with the digits 0–9? (a) 50–100 million (b) 50–100 billion (c) 50–100 trillion
   ANS: $36^7 = 78\,364\,164\,096 \approx \boxed{\text{50–100 billion}}$ or $\boxed{\text{(b)}}$

7. What integer is closest to the sum $\frac{7}{3} + \frac{3}{7}$?
   ANS: $\frac{7}{3} + \frac{3}{7} = \frac{58}{21} = 2.761\,9$ so nearest integer is $\boxed{3}$

8. A teacher wants to curve a set of test grades so that the student who made the high score of 90 gets 100, and the student who made 66 will get an 80. The teacher uses a linear function. What is the new grade for a student whose old grade was 51?
   ANS: If $f(x) = 100 + \frac{100-80}{90-60}(x - 90)$, then $f(51) = \boxed{74}$

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9. The book, *Alan Turing: The Enigma*, inspired a play that ran in both London and New York City. What is the name of the play?

   ANS: $\boxed{\text{Breaking the Code}}$

10. Three unit circles are mutually tangent and enclose a triangular region R, which is bounded by three arcs of the three circles. Find the area of R.

    ANS: area $\triangle = \sqrt{3}$, sector of $\bigcirc = \pi/6$, net $\sqrt{3} - 3(\pi/6) = \boxed{\sqrt{3} - \frac{\pi}{2}}$

11. What number is halfway between $\frac{1}{3}$ and $\frac{3}{4}$?

    ANS: $\frac{\frac{1}{3}+\frac{3}{4}}{2} = \boxed{\frac{13}{24}}$

12. Five straight lines are drawn in the plane. What is the largest possible number of points of intersection?

    ANS: $\binom{5}{2} = \boxed{10}$

13. Write the next term of the sequence that begins: 5,12,20,29,39,...

    ANS: $5 \overbrace{\quad}^{7} 12 \overbrace{\quad}^{8} 20 \overbrace{\quad}^{9} 29 \overbrace{\quad}^{10} 39 \overbrace{\quad}^{11} \boxed{50}$

14. A triangle has sides of length 5, 6, and 7. Find the area of the triangle.

    ANS: $Area = \sqrt{9(9-5)(9-6)(9-7)} = \boxed{6\sqrt{6}}$

15. A square is inscribed in a circle, which in turn is inscribed in a square. What percentage of the area of the large square is inside the small square?

    ANS: $\boxed{50}$ %

Tiebreaker Question

Arrange the following three numbers from smallest to largest: $\frac{1}{8}$, $\frac{1}{9}$, or $\frac{2}{17}$.

ANS: $\boxed{\frac{1}{9}} = 0.111\,111\,111 < \boxed{\frac{2}{17}} = 0.117\,647\,058 < \boxed{\frac{1}{8}} = 0.125$

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1. If $\sin x = \frac{1}{4}$ and $0 < x < \pi/2$, what is $\sin 2x$?

   ANS: $\sin 2x = 2\sin x\cos x = 2\cdot\frac{1}{4}\sqrt{1-\frac{1}{16}} = \boxed{\frac{1}{8}\sqrt{15}}$

2. CAS describes a type of computer software. What do the letters stand for?

   ANS: Computer Algebra System

3. What is the smallest prime between 200 and 300?

   ANS: $201 = 3\times 67$, $203 = 7\times 29$, $205 = 5\times 41$, $207 = 3^2 23$, $209 = 11\times 19$, $211 = \boxed{211}$

4. If g is a function such that $g(1) = 3$, $g(2) = -1$, and

   $$g(n+1) = g(n) + 2g(n-1)$$

   for $n \geq 2$, what is $g(4)$?

   ANS: $g(3) = g(2) + 2g(1) = -1 + 6 = 5$, $g(4) = g(3) + 2g(2) = 5 - 2 = \boxed{3}$

5. The graph of a cubic polynomial has x-intercepts at -1, 0 and 1. What is a possible expression for the polynomial?

   ANS: $\boxed{x\,(x+1)\,(x-1)}$ or $\boxed{x^3 - x}$ (or a nonzero multiple)

6. Factor completely the integer $7^3 + 8^3$.

   ANS: $7^3 + 8^3 = (7+8)\,(7^2 - 7\cdot 8 + 8^2) = 3\times 5\times 57 = \boxed{3^2 5\times 19}$

7. Eight poker players are seated at a round table. Up to rotation of the table and chairs, how many rearrangements are possible? (Because of the betting order, Arlene on your left is different from Arlene on your right.)

   ANS: $7! = \boxed{5040}$

8. If 108 centimeters of wire are used to build the skeleton of edges of a cube, what is the volume of the cube?

   ANS: $84/12 = 7$ and $7^3 = \boxed{343}$

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9. What is the smallest integer whose square is less than 200?

   ANS: $\boxed{-14}$ since $(-15)^2 = 225$, $(-14)^2 = 196$

10. What is the volume of the largest cube that can fit inside a sphere of radius $\sqrt{3}$?

    ANS: $Volume = 2^3 = \boxed{8}$

11. For what choices of a does the polynomial $9a - 12x + 4x^2$ have exactly one real root?

    ANS: $a = \boxed{1}$ since $9 - 12x + 4x^2 = (2x - 3)^2$

12. Rene Descartes (1596–1650) is best remembered for making connections between A) trigonometry and logarithms, B) algebra and geometry, or C) statistics and combinatorics.

    ANS: $\boxed{\text{B) algebra and geometry}}$

13. If $\sin(2\theta) = 1$, and θ is in the first quadrant, what is $\tan(\theta)$?
    ANS: 1 If $\sin 2\theta = 1$, then $2\theta = \pi/2$, so $\theta = \pi/4$ and $\tan(\pi/4) = \boxed{1}$

14. Solve the equation $x\sqrt{.04} = 7$

    ANS: $x = \frac{7}{\sqrt{.04}} = \boxed{35}$

15. What is the 17th term of the arithmetic sequence that begins 5, 9, 13, ...?

    ANS: $\boxed{65}$ $5 + 4 \cdot 16 = \boxed{69}$

Tiebreaker Question

How many distinct real roots does $x^2 - 2x + 1$ have?
ANS: $\boxed{\text{one}}$ $x^2 - 2x + 1 = (x - 1)^2$

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1. Write the expression $\frac{4x+5}{(x+2)(2x+1)}$ in partial fraction form (i.e. $\frac{A}{Bx+C}+\frac{D}{Ex+F}$, where A, B, C, D, E, and F are integers).

   ANS: $\frac{4x+5}{(x+2)(2x+1)} = \boxed{\frac{1}{x+2}+\frac{2}{2x+1}}$

2. If $\cos(3\theta)=0$, and θ is in the first quadrant, what is $\sin(\theta)$?
   ANS: $\boxed{1/2}$

3. What is the largest 3-digit prime?

   ANS: $\boxed{997}$

4. What is the midpoint of the line segment joining $(14,-4)$ and $(-2,10)$?

   ANS: $\left(\frac{14-2}{2}, \frac{-4+10}{2}\right) = \boxed{(6,3)}$

5. The formula $e^{i\pi}+1=0$ relates five of the most famous numbers in mathematics. What is $e^{2i\pi}$?

   ANS: $e^{2i\pi} = \boxed{1}$

6. Sarah is a 75% free throw shooter. What is her expected score if she shoots a one and one (if she makes the first she gets a second chance)?

   ANS: $1 \times \frac{3}{4} \times \frac{1}{4} + 2 \times \frac{3}{4} \times \frac{3}{4} = \boxed{\frac{21}{16}}$

7. The 8 faces of a regular octahedron are equilateral triangles. How many vertices does a regular octahedron have?

   ANS: $\frac{8\times 3}{4} = \boxed{6}$

8. This mathematician first proved that if p is a prime, then p evenly divides the number 2^p-2. The name of this mathematician is A) Fermat, B) Gauss, or C) Euler.

   ANS: $\boxed{\text{A) Fermat}}$ (special case of Fermat's Little Theorem)

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9. What is the coefficient of x^3y^3 in the expansion of $(x+y)^6$?

   ANS: $(x+y)^6 = x^6 + y^6 + 6xy^5 + 6x^5y + 15x^2y^4 + \boxed{20}\, x^3y^3 + 15x^4y^2$ or $\binom{6}{3} = \boxed{20}$

10. What is the sum of the binomial coefficients (6 choose zero) plus (6 choose 1) plus $\cdots$ plus (6 choose 6)?

    ANS: $\boxed{64} = 2^6 = (1+1)^6 = \sum_{i=0}^{6} \binom{6}{i}$

11. According to Descartes' Rule of Signs, how many positive real roots does the polynomial equation $3x^4 + 10x^2 + 5x + 4 = 0$ have?

    ANS: $\boxed{\text{None}}$ (since there are no sign changes)

12. Bo is going to the store to buy candy that will cost somewhere between 5 cents and 36 cents. What is the fewest number of coins Bo can carry in order to be certain to have exact change to buy the candy?

    ANS: 1¢, 1¢, 1¢, 1¢, 5¢, 10¢, 10¢, 10¢ $\boxed{8}$ coins

13. Out of a set of eleven natural-number scores on a 20-point quiz, the mean is 12, the median is 12, and the mode is 10. Find the maximum number of perfect scores possible on this test.

    ANS: $\boxed{3}$

14. In the geometric progression 189, -63, 21, n, what is the value of n?

    ANS: $\boxed{-7} = 21/(-3)$

15. A computer sequentially computes integers by the following rule: If n is odd then replace n by $3n+1$; otherwise replace n by $n/2$. If n starts at 6, what is n after 5 iterations?

ANS: $6 \rightarrow 3 \rightarrow 10 \rightarrow 5 \rightarrow 16 \rightarrow \boxed{8}$

Tiebreaker Question

What is the prime factorization of 105?
ANS: $105 = \boxed{3 \times 5 \times 7}$

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1. According to Descartes' Rule of Signs, how many negative real roots does the polynomial equation $x^4 - 8x^2 + 6x - 7 = 0$ have?

   ANS: $\boxed{\text{one}}$ (since there is one sign change if x is replaced by $-x$)

2. What is the smallest number of 3-inch by 5-inch rectangular tiles required to form a square with no cutting or gaps?

   ANS: $\boxed{15}$

3. What is the converse of the statement, "If roses are red, then violets are blue"?

   ANS: $\boxed{\text{If violets are blue, then roses are red.}}$

4. If the first term of a geometric sequence is $\frac{2}{3}$ and the second term is $\frac{1}{3}$, what is the fifth term?

   ANS: $\frac{2}{3} \times \left(\frac{1}{2}\right)^4 = \boxed{\frac{1}{24}}$

5. What is the perimeter of a regular hexagon inscribed in a circle of radius 7?

   ANS: $\boxed{42}$

6. Every mathematician has an Erdös number (which might actually be $+\infty$). How is this number defined?

   ANS: Paul Erdös has an Erdös number of 0. Those who published a joint paper with Erdös have an Erdös number of 1. Those who have a joint paper with someone with Erdös number n has an Erdös number $\leq n+1$. The Erdös number is the length of the shortest such path.

7. What is the smallest positive integer divisible by 1, 2, 3, 4, 5, and 6?

   ANS: $2^2 \times 3 \times 5 = \boxed{60}$

8. On a graphing calculator, a "complete graph" of a function displays all the interesting, important, or distinguishing features of the graph. What is the largest number of maximum and minimum points (that is, high and low points) one could see in a complete graph of a polynomial of degree 5?

   ANS: $\boxed{4}$

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9. A regular icosahedron has 20 faces, each of which is an equilateral triangle. If the midpoint of each face is connected with an edge to the midpoint of each adjacent face, what solid do these new edges determine?

   ANS: $\boxed{\text{Dodecahedron}}$ (12 faces, each face is a pentagon.)

10. The sum of 4 consecutive integers is 82. What is the smallest of the 4 integers?

    ANS: $\boxed{19} + 20 + 21 + 22 = 82$

11. Let $x_n = 6^n + n$. What is the last digit of x_{14}?
    ANS: $\boxed{0}$ since $6^{14} + 14 = 78\,364\,164\,110$

12. Eight people are related to each other according to the graph of the edges of a cube. What is the largest number of unrelated people that can be formed?
    ANS: $\boxed{4}$

13. Give 3 distinct decimal digits summing to 23.
    ANS: $\boxed{6,\ 8,\ 9}$

14. A computer sequentially computes integers by the following rule: If n is a perfect square, then multiply by 2; otherwise subtract 7. Starting at $n = 23$, what is the integer after 6 iterations?

    ANS: $23 \to 16 \to 32 \to 25 \to 50 \to 43 \to \boxed{36}$

15. The probability of picking a dog to finish in the top 3 at the dog track is $\frac{1}{3}$. What is the probability of picking 3 straight losers?

    ANS: $\boxed{\frac{8}{27}} = \left(\frac{2}{3}\right)^3$

Tiebreaker Question

How many real roots does $x^3 - x^2 + x - 1$ have?
ANS: $\boxed{\text{one}}$ $x^3 - x^2 + x - 1 = (x^2 + 1)(x - 1)$

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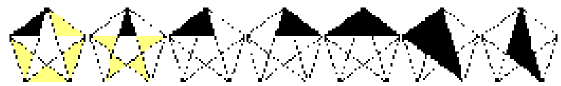
1. Draw a pentagon with each of its five diagonals. How many triangles are contained in the figure?

   ANS: $\boxed{35}$ 5 each of types

2. What is the sum of the roots of the polynomial $x^2 + 3x - 18$?

   ANS: $\boxed{-3}$ since $(x-a)(x-b) = ab - (a+b)x + x^2$ (Note that $x^2 + 3x - 18 = (x+6)(x-3)$ has roots -6 and 3)

3. How many numbers between 386 and 724 are divisible by 17?

   ANS: $17 \times 23 = 391$, $17 \times 42 = 714$, $42 - 23 + 1 = \boxed{20}$

4. Sarah is a 95% free throw shooter. What is the probability that she misses 2 in a row?

   ANS: $\left(\frac{1}{20}\right)^2 = \boxed{\frac{1}{400}} = \boxed{0.002\,5} = \boxed{0.25}\,\%$

5. What is the median of 5.7, 8.3, 7.1, 4.8, 6.8, 8.5?

   ANS: $\dfrac{6.8 + 7.1}{2} = \boxed{6.95}$

6. A biological brick grows 20% in length, 5% in width, and shrinks in height by 20%. Is it larger or smaller than when it started out?

   ANS: $\boxed{\text{larger}}$ since $1.20 \times 1.05 \times 0.8 = 1.008$

7. In 1984, Louis de Brange solved which of the following famous problems: A) Continuum Hypothesis, B) Fermat's Last Theorem, or C) Bieberbach Conjecture?

   ANS: $\boxed{\text{C)}}$ or $\boxed{\text{Bieberbach Conjecture}}$

8. If g is a function such that $g(1) = 1$, $g(2) = -1$, and $g(n+1) = 2g(n) + g(n-1)$ for $n \geq 2$, what is $g(4)$?

   ANS: $g(3) = 2g(2) + g(1) = -2 + 1 = -1$ $\quad g(4) = 2g(3) + g(2) = -2 + (-1) = \boxed{-3}$

Team Competition
1:35

9. What is the smallest integer greater than the sum $\frac{17}{7} + \frac{7}{17}$?

   ANS: $\frac{17}{7} + \frac{7}{17} = 2.84035 = \boxed{3} - 0.159\,66$

10. How many ways can the top 3 finishers be picked in a 8-person race?

    ANS: $\boxed{336} = 8 \times 7 \times 6$

11. What is the sum of the first 20 even integers: 2, 4, 6, $\cdots$, 40?

    ANS: $\sum_{k=1}^{20} 2k = \boxed{420} = 2 \times \frac{20 \times 21}{2}$

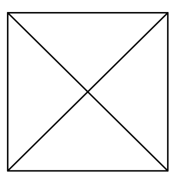
12. If there are 60 seconds in a minute, 60 minutes in an hour, and 24 hours in a day, how many seconds were there yesterday?

    ANS: $24 \times 60 \times 60 = \boxed{86\,400}$ seconds

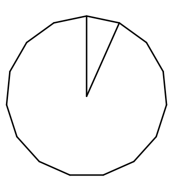
13. Which regular polyhedron has the same number of faces as vertices?

    ANS: $\boxed{\text{Tetrahedron}}$ 4 faces and 4 vertices

14. Draw two line segments, each 6 cm in length, intersecting at their midpoints. Connect each endpoint to each adjacent endpoint to form a quadrilateral. What is the maximum number of squares centimeters possible in the area of the quadrilateral?

    ANS: $\boxed{18} = \left(\frac{6}{\sqrt{2}}\right)^2$ 

15. If the interior angle of a regular polygon is $156\,^\circ$, how many sides does it have?

ANS:  $\frac{360^\circ}{24^\circ} = \boxed{15}$ sides

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Tiebreaker Question

What is the prime factorization of 91?
ANS: $91 = \boxed{7 \times 13}$

CSU Math Day 2005
Team Competition
1:50

1. Neglecting order of addition, in how many ways can 37 be written as a sum of 3 distinct primes?

   ANS: $\boxed{5}$ ways since $37 = 29+5+3 = 23+11+3 = 19+13+5 = 19+11+7 = 17+13+7$

2. In an algebra class of 20 students, each student shakes hands with each of the other students exactly once. How many handshakes are there?

   ANS: $\binom{20}{2} = \dfrac{20 \times 19}{2} = \boxed{190}$

3. Inspired by his work with Emmy Noether, Bartel van der Waerden wrote one of the early textbooks in the area of A) numerical analysis, B) differential equations, or C) modern algebra

   ANS: $\boxed{\text{C)}}$ or $\boxed{\text{modern algebra}}$

4. Two baseball teams, the Green Socks and the Yellow Elbows, start a 7-game series. For each game, the odds are 3 to 1 in favor of the Green Socks. What is the probability that the Green Socks win in five games?

   ANS: $GGGYG + GGYGG + GYGGG + YGGGG = 4 \times \left(\frac{3}{4}\right)^4 \times \frac{1}{4} = \boxed{\frac{81}{256}}$

5. What is the smallest positive integer whose cube is larger than 1693?

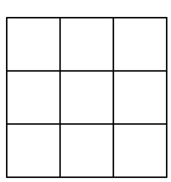
   ANS: $11^3 = 1331$, $12^3 = 1728$, so $\boxed{12}$

6. List the following three numbers in increasing order: $\sqrt{5}$, $\sqrt[3]{11}$, $\sqrt[6]{123}$

   ANS: $\boxed{\sqrt[3]{11} < \sqrt[6]{123} < \sqrt{5}}$ since $\left(\sqrt[3]{11}\right)^6 = 121 < 123 = \left(\sqrt[6]{123}\right)^6 < 125 = \left(\sqrt{5}\right)^6$

7. How many ways can you insert a cube with side length five into a square hole with side length five?
   ANS: $\boxed{24} = 6 \times 4$

8. A square of side length 3 is tiled into 1×1 subsquares. How many rectangles are there in the figure?

   ANS: $\boxed{36}$  $9\,(1 \times 1) + 6\,(1 \times 2) + 6\,(2 \times 1) + 3\,(1 \times 3) + 3\,(3 \times 1) +$

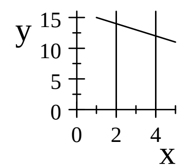
   $4\,(2 \times 2) + 2\,(2 \times 3) + 2\,(3 \times 2) + 1\,(3 \times 3)$

Team Competition
1:50

9. What is the first digit of $2^{33} \cdot 5^{32} - 2$?
   ANS: $2^{33} \cdot 5^{32} - 2 = 2 \cdot 10^{32} - 2 = \boxed{1}99\,999\,999\,999\,999\,999\,999\,999\,999\,999\,998$

10. What is the area of the region bounded by the x-axis, the vertical lines $x = 2$, $x = 4$, and the line $y = -x + 16$?

    ANS: $\boxed{26} = width \times average\ height = 2 \times 13$



11. A sweater was originally marked up 50% and then down 20% to \$60.00. What was the original price?

    ANS: $\$\boxed{50} = \dfrac{60.0}{1.5 \cdot 0.8}$

12. The equation $x_{n+1} = \frac{1}{2}\left(x_n + \frac{3}{x_n}\right)$ provides an algorithm for approximating $\sqrt{3}$. Starting with $x_1 = 1$, what is x_3 (as a rational number)?

    ANS: $x_2 = \frac{1}{2}(1+3) = 2$, $x_3 = \frac{1}{2}\left(2 + \frac{3}{2}\right) = \boxed{\frac{7}{4}} = \boxed{1\frac{3}{4}}$

13. Sally invested \$1000, part at 10% and the rest at 5%. If the annual interest income from both investments was \$90.00, how much was invested at 10%?

    ANS: $x \cdot 0.1 + (1000 - x) \cdot 0.05 = 90$, Solution is: $\$\ \boxed{800.00}$

14. What is 80° equal to in radians?

    ANS: $\frac{4}{9}\pi$

15. What is the smallest positive integer that can be written as the sum of 2 cubes in 2 different ways."

    ANS: 1729 ($= 12^3 + 1^3 = 10^3 + 9^3$) (Credited to the Indian mathematician Ramanujan)

Tiebreaker Question

Which number is smaller, $2 + 3\sqrt{5}$ or $2 + 5\sqrt{3}$?
ANS: $\boxed{2 + 3\sqrt{5}} = 8.708\,2 < 2 + 5\sqrt{3} = 10.66$

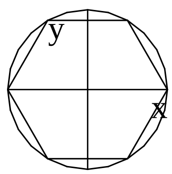
CSU Math Day 2005
Team Competition
2:05

1. Find C so that the equation $x^2 - 20x + C = 0$ has exactly one real root.

   ANS: When $C = \boxed{100}$, $x^2 - 20x + 100 = (x - 10)^2$

2. A regular hexagon is inscribed in a circle of radius 10 cm. What is the perimeter of the hexagon?

   ANS: $\boxed{60}$ cm



3. Solve the system of equations

$$\begin{aligned} y + z &= 1 \\ x + z &= 1 \\ x + y &= 1 \end{aligned}$$

   ANS: By symmetry, $\boxed{x = y = z = \frac{1}{2}}$

4. In 1962 a mathematician named Edward O. Thorp wrote a book entitled *Beat the Dealer* that claimed to give a winning strategy, based upon large-scale computer simulations, for a certain card game. What is the name of that card game?

   ANS: $\boxed{\text{Blackjack}}$ or $\boxed{21}$

5. What is the area of a regular octagon of edge 1?

   ANS: Large $\square -$ four corners $= \left(1 + 2\left(\frac{1}{\sqrt{2}}\right)\right)^2 - 4\left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right)^2 = \boxed{2 + 2\sqrt{2}}$

6. What is the smallest value of the expression $x + \frac{1}{x}$ if x is a positive real number?

   ANS: $\left[x + \frac{1}{x}\right]_{x=1} = \boxed{2}$

7. What complex numbers are equal to their squares?

   ANS: $x = x^2$, Solution is: $x = \boxed{0}$, $x = \boxed{1}$

8. If 2 is the first prime, what is the fifteenth prime?

   ANS: $2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, \boxed{47}$

Team Competition
2:05

9. Find the smallest palindromic cube larger than ten.
   ANS: $\boxed{343} = 7^3$

10. Name the five Platonic solids (regular polyhedra).

    ANS: Tetrahedron, cube (or hexahedron), octahedron, dodecahedron, icosahedron

11. Three distinct lines in a plane may intersect in k points. What is the sum of all possible values of k?

    ANS: $\boxed{6}$

12. The formula $e^{i\pi} + 1 = 0$ relates five of the most popular numbers in mathematics. What is π rounded to 10 significant digits?

    ANS: $\pi = 3.14159\,26535\,8979\ldots \approx \boxed{3.14159\,2654}$

13. Find a solution to the equation $p^a = q^b + 1$ where p and q are prime and a and b are integers greater than one.
    ANS: $\boxed{3^2 = 2^3 + 1}$

14. A committee of 3 people is to be chosen from among 4 men and 6 women. How many ways can this be done if the committee must include at least one man and at least one woman?

    ANS: $\boxed{96} = \binom{10}{3} - \binom{4}{3} - \binom{6}{3}$

15. A Social Security number has nine digits. Assuming the digits are random, what is the expected number of fives in a social security number?

    ANS: $9 \times \frac{1}{10} = \boxed{\frac{9}{10}}$

Tiebreaker Question

Which number is largest: $5 + 6\sqrt{7}$, $6 + 7\sqrt{5}$, or $7 + 5\sqrt{6}$?
ANS: $\boxed{6 + 7\sqrt{5}} \approx 21.652 > 5 + 6\sqrt{7} \approx 20.875 > 7 + 5\sqrt{6} \approx 19.247$

CSU Math Day 2005
Team Competition
2:15

1. If a cube has a volume of $125\,\text{cm}^3$, what is its surface area?

   ANS: $\boxed{150}$ cm^2

2. If $2^x = 7$, then what is 2^{3x+2}?

   ANS: $2^{3x+2} = (2^x)^3\, 2^2 = 7^3 2^2 = \boxed{1372}$

3. What is the sum of all the integers greater than 5 and less than 25?

   ANS: $\sum_{n=6}^{24} n = \frac{24\cdot 25}{2} - \frac{5\cdot 6}{2} = \boxed{285}$

4. Henry explained his age by saying, "$\frac{2}{5}$ of my age less $\frac{1}{9}$ of what it will be a year from now is equal to $\frac{1}{3}$ of what my age was 5 years ago." What is his age now?

   ANS: $\frac{2}{5}a - \frac{1}{9}(a+1) = \frac{1}{3}(a-5)$, Solution is : $a = \boxed{35}$

5. If two cards are drawn from a standard deck of 52 cards, what is the probability that both are kings?

   ANS: $\frac{\binom{4}{2}}{\binom{52}{2}} = \frac{6}{1326} = \boxed{\frac{1}{221}}$

6. A 6-foot man casts a 7-foot shadow. A flag pole next to him casts a 22-foot shadow. How tall in the flag pole?

   ANS: $\frac{6}{7} = \frac{x}{22}$, Solution is: $\boxed{\frac{132}{7}}$ or $\boxed{18\frac{6}{7}}$

7. What is the area of the right triangle whose hypotenuse is 17 if one of the legs has length 15?

   ANS: $x^2 + 15^2 = 17^2$, Solution is: 8 so $\frac{1}{2} \times 8 \times 15 = \boxed{60}$

8. On a certain planet, a stone falls (from rest) 9 feet in 2 seconds, how far will it fall in 4 seconds?

   ANS: $2^2 g = 9$, Solution is: $g = \frac{9}{4}$, so $4^2 \times \frac{9}{4} = \boxed{36}$ ft

Team Competition
2:15

9. Movie phone numbers always begin with the first three digits 555. How many movie phone numbers have four distinct last digits?
   ANS: $\boxed{5040} = 10 \cdot 9 \cdot 8 \cdot 7$

10. Completely factor the polynomial $6x^3 - 7x^2 - 7x + 6$.
    ANS: $\boxed{(2x-3)(3x-2)(x+1)}$

11. The distance between the points $(-5, 4)$ and $(7, c)$ is 13. Find all the possible values for c.
    ANS: $(-5-7)^2 + (4-c)^2 = 13^2$, Solution is : $\boxed{\{c=-1\}, \{c=9\}}$

12. Early in his career, John Kemeny helped Albert Einstein with his mathematics. Later, he and co-developer Thomas Kurtz wrote the computer language A) FORTRAN, B) Pascal, or C) BASIC
    ANS: $\boxed{\text{C) BASIC}}$

13. If it takes 27 minutes to inflate a large beach ball to a radius of 3 feet, how long will it take to inflate a larger beach ball to a radius of 5 feet?
    ANS: $\boxed{125}$ min

14. Four positive integers have a sum of 14. What is the minimum possible value for the sum of their squares?
    ANS: $\boxed{50} = 3^2 + 3^2 + 4^2 + 4^2$

15. Express as an integer: $\sqrt{\sqrt{\sqrt{256}}}$
    ANS: $\boxed{2} = \sqrt{\sqrt{\sqrt{256}}}$

Tiebreaker Question

Which number is largest: $3 + 5\sqrt{6}$, $5 + 6\sqrt{3}$, or $6 + 3\sqrt{5}$?
ANS: $5 + 6\sqrt{3} = 15.392 > 3 + 5\sqrt{6} = 15.247 > 6 + 3\sqrt{5} = 12.708$

CSU Math Day 2005
Team Competition
Small School Final

1. If $f(x) = x^2 - x + 1$, what is $f(f(f(-1)))$?

   ANS: $f(-1) = 3,\ f(3) = 7,\ f(7) = \boxed{43}$

2. What is the perimeter of an isosceles triangle with base 12 and area 54?

   ANS: $\boxed{12 + 6\sqrt{13}}$

3. John and Jill traded positions several times while rowing a canoe through the Boundary Waters of Minnesota. With Jill at the rear, the canoe went fast enough to complete the entire trip in 10 hours, and with John at the rear the trip would have taken 14 hours. The trip actually took 12 hours. For how many hours did Jill sit in the back of the canoe?

   ANS: $\boxed{5}$ hours

4. If the absolute value of $2x + 1$ is equal to the absolute value of $2x - 1$, what is x?

   ANS: $x = \boxed{0}$

5. How many primes are there between 200 and 210?

   ANS: $\boxed{\text{None}}$ or $\boxed{\text{Zero}}$ since $201 = 3 \times 67,\ 203 = 7 \times 29,\ 205 = 5 \times 41,\ 207 = 3^2 23,\ 209 = 11 \times 19$

6. Alice Tallwalker plans to walk around the Earth at the equator. Assuming this is possible (she walks on water) and that the Earth is a perfect sphere, how much farther will her nose travel than her feet? Alice holds her nose 6 feet above the ground when walking.

   ANS: $2\pi(r + 6) - 2\pi r = \boxed{12\pi}$ ft

7. List the following three numbers in increasing order: π, $22/7$, and $355/113$? Hint: $355/113 \approx 3.141\,592\,92$.

   ANS: $\boxed{\pi} \approx 3.141592654 < \boxed{355/113} = 3.141\,6 < \boxed{22/7} \approx 3.142\,9$

8. In a lab, 448 grams of a radioactive material with a half-life of 4 months is placed in a lead container. How many grams are left after 2 years?
   ANS: $448 \times \left(\frac{1}{2}\right)^6 = \boxed{7}$ grams

Team Competition
Small School Final

9. A multiple-choice exam has 16 questions with 4 choices per question. If you answer the questions randomly, what is the expected number of correct responses?

   ANS: $16 \cdot \frac{1}{4} = \boxed{4}$

10. What is the smallest positive integer that is both a perfect square and a perfect cube?
    ANS: $\boxed{1} = 1^2 = 1^3$

11. Two circles are mutually tangent at one point, and the smaller circle passes through the center of the larger circle. What is the ratio between the areas of the two circles?

    ANS: $\boxed{4:1}$ or $\boxed{1:4}$

12. Pascal's Pizza Parlor is changing the size (diameter) of its circular pizza from 12 inches to 15 inches and increasing the number of slices per pizza for 6 to 8 congruent pieces. What is the percentage increase, to the nearest whole percent, of the size of each new slice?

    ANS: $\boxed{17}$ % since $\frac{15^2}{8} \div \frac{12^2}{6} = \frac{75}{64} = 1.1719$

13. If $6^p = 11$, then what is 216^p?

    ANS: $216^p = (6^3)^p = (6^p)^3 = 11^3 = \boxed{1331}$

14. What is the coefficient of x^3 in the expansion of $(2x+1)^7$?

    ANS: $(2x+1)^7 = 128x^7 + 448x^6 + 672x^5 + 560x^4 + \boxed{280}x^3 + 84x^2 + 14x + 1$ or $2^3\binom{7}{3} = 280$

15. The divergent series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ has a musical name. What is that name?

    ANS: Harmonic Series

Tiebreaker Question

Factor 2005 as a product of primes.
ANS: $2005 = \boxed{5 \times 401}$

CSU Math Day 2005
Team Competition
Small School Final (Round 2)

1. If $7^a = 16$, what is $7^{0.75a}$?

   ANS: $7^{0.75a} = (7^a)^{3/4} = (16)^{3/4} = \boxed{8}$

2. What is the least common multiple of 119 and 147?

   ANS: $\text{lcm}\,(119, 147) = \boxed{2499}$ Note that $119 = 7 \times 17$ and $147 = 3 \times 7^2$, so $\text{lcm}\,(119, 147) = 3 \times 7^2 \times 17 = 2499$

3. Name the largest positive integer whose forth power is less than 2000.

   ANS: $\boxed{6}$ since $6^4 = 1296$, $7^4 = 2401$

4. The average of 4 numbers is 28. What is the average of these 4 numbers together with 23?

   ANS: $\dfrac{4 \times 28 + 23}{5} = \boxed{27}$

5. The set S has 8 elements. If a and b are (distinct) elements of S, how many subsets of S contain both a and b?

   ANS: $\boxed{64} = 2^6$

6. List the following three numbers in increasing order: $x = 1 + 2 + 3 + \cdots + 37$, $y = 1 \times 2 \times 3 \times 4 \times 5 \times 6$, and $z = 3^6$.

   ANS: $\boxed{x} = \sum_{i=1}^{37} i = 703 < \boxed{y} = 6! = 720 < \boxed{z} = 3^6 = 729$

7. What is the smallest positive integer n such that $n^2 - n + 17$ is not prime?

   ANS: $\boxed{17}$, since

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
17	19	23	29	37	47	59	73	89	107	127	149	173	199	227	257	17^2

8. Find a two-digit number ending in 6 whose square has the final two digits equal to the original number.

   ANS: $\boxed{76}$ $76^2 = 5776$ $(x6)^2 = (10x + 6)^2 = 100x^2 + 120x + 36$, only solution is $x = 7$

Team Competition
Small School Final (Round 2)

9. The graph of a cubic polynomial crosses the x-axis at $x = -2$, $x = 1$, and $x = 3$. In expanded form, what is one such polynomial?

   ANS: $(x+2)(x-1)(x-3) = \boxed{x^3 - 2x^2 - 5x + 6}$ or some nonzero multiple thereof

10. An octahedron is a regular solid whose 8 faces are equilateral triangles. What is the distance between a pair of opposite vertices, assuming each edge of the octahedron is of length $\sqrt{2}$?

    ANS: $\boxed{2}$

11. Given an integer n with $0 < n < 23$, what is the remainder when n^{22} is divided by 23?

    ANS: $\boxed{1}$ (Fermat's Little Theorem) $2^{22} \bmod 23 = 1$, $3^{22} \bmod 23 = 1$, $5^{22} \bmod 23 = 1$, etc.

12. Stefan Banach's work that included the theory of Banach spaces is sometimes said to mark the birth of A) group theory, B) statistics, or C) functional analysis.

    ANS: $\boxed{\text{C) functional analysis}}$

13. Art, Betty and Claude are now 17, 19, and 23 years old. How many years will it be until the next time they have prime-numbered ages?

    ANS: $\boxed{\text{24 years from now}}$ they will have ages $41, 43, 47$ (Twin primes are $29, 31$; $41, 43$)

14. How far is the point (3,0) from the line $3y = 4x$?

    ANS: $\frac{x}{3} = \frac{4}{5} \Longrightarrow x = \boxed{\frac{12}{5}} = \boxed{2.4}$

15. What is the radius of the sphere whose volume is 14 times its surface area?

    ANS: $\frac{4}{3}\pi r^3 = 14 \times 4\pi r^2$, Solution is: $\boxed{42}, 0$

Tiebreaker Question

What is the prime factorization of 143?
ANS: $143 = \boxed{11 \times 13}$

CSU Math Day 2005
Team Competition
Large School Final

1. During a recent election, Alfie, Betty, and Gammer received votes for mayor. Alfie received $\frac{1}{3}$ as many votes as Betty and 2 times as many as Gammer. If the total number of votes was 27 000, how many did each person get?

   ANS: $\boxed{\text{Alfie: 6000 Betty: 18 000 Gammer: 3000}}$

2. The sum of 6 consecutive integers is 153. What is the largest of the 6 integers?

   ANS: $\boxed{28}$ $\sum_{k=23}^{28} k = 153$

3. How many real roots does the polynomial equation $x^5 + x^4 - x^2 - x = 0$ have?
   ANS: $\boxed{\text{Three}}$ real: $x(x-1)(x+1)(x^2+x+1) = x^5 + x^4 - x^2 - x$

4. Six straight lines are drawn in the plane. What is the largest possible number of points of intersection?

   ANS: $\binom{6}{2} = \boxed{15}$

5. Give an equation in the form $ax + by = c$, where a, b, and c are integers, for the line through $(4, -4)$ that is parallel to the line $y = \frac{2}{3}x - \frac{5}{3}$.

   ANS: $\boxed{2x - 3y = 20}$ or $\boxed{-2x + 3y = -20}$ $y + 4 = \frac{2}{3}(x - 4) \Longrightarrow 3y + 12 = 2x - 8$

6. If the absolute value of $x + 1$ is equal to the absolute value of x, what is x?

   ANS: $|x + 1| = |x|$, Solution is: $x = \boxed{-\frac{1}{2}}$

7. The 60 rearrangements of the word 'SARAH' (spelled S-A-R-A-H) are listed in alphabetical order. Where does SARAH appear on this list?

   ANS: $\boxed{\text{53rd}}$ SAAHR^{49} SAARH^{50} SAHAR^{51} SAHRA^{52} SARAH^{53} SARHA^{54} SHAAR^{55} SHARA^{56} SHRAA^{57} SRAAH^{58} SRAHA^{59} SRHAA^{60}

Team Competition
Large School Final

8. A pyramid is built out of blocks by placing 121 blocks on the floor, placing 100 blocks on top of the bottom layer, and so forth. How many cubes are there in the pyramid?

   ANS: $\sum_{i=1}^{11} i^2 = \boxed{506} = \frac{11 \cdot 12 \cdot 23}{6}$

9. What is the decimal representation of the base 7 number 234_7?

   ANS: $2 \times 7^2 + 3 \times 7 + 4 = \boxed{123}$

10. The number of times a pendulum oscillates in a given time varies inversely as the square root of its length. If a 40 inch pendulum oscillates once per second, what is the length of a pendulum that oscillates once every three seconds?

    ANS: $\boxed{360}$ in

11. If $f(x) = 4x - 5$, what is $f\left(f(f(\frac{1}{2}))\right)$?

    ANS: $f\left(\frac{1}{2}\right) = -3$, $f(-3) = -17$, $f(-17) = \boxed{-73}$

12. In a certain set of base-ten addition problems, when a two-digit number of the form ab is added to a one-digit number b, the result is a three-digit number with ones digit c. How many triples (a, b, c) satisfy these conditions?

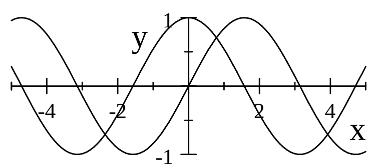
    ANS: $\boxed{5}$ $(9,5,0)$, $(9,6,2)$, $(9,7,4)$, $(9,8,6)$, $(9,9,8)$ (Note that c is irrelevant.)

13. George is a 90% free-throw shooter. What is his expected score if he shoots a one-and-one? (If he hits the first, he has a chance for a second.)

    ANS: $1 \times \frac{9}{10} \times \frac{1}{10} + 2 \times \frac{9}{10} \times \frac{9}{10} = \boxed{\frac{171}{100}}$

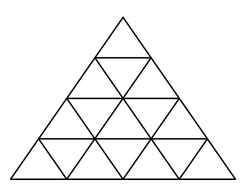
14. Give all solutions to the equation $\sin(x) = \cos(x)$.

    ANS: $x = \boxed{\frac{\pi}{4} + \pi n,\ n \text{ an integer}}$



15. A wooden tetrahedron of edge 4 inches is painted red. The tetrahedron is then cut into 64 tetrahedra of edge length 1 by making 12 saw cuts. How many of the small tetrahedra have exactly 1 red face?

ANS: 4 faces $\times$ 7 tetrahedra per face $= \boxed{28}$ tetrahedra



Tiebreaker Question

List the following in increasing order: $x = 10 + 11 + 12 + \cdots + 23$, $y = 15^2$, $z = 3 \cdot 4 \cdot 5 \cdot 6$

$\boxed{y} = 15^2 = 225 < \boxed{x} = \sum_{k=10}^{23} k = 231 < \boxed{z} = 3 \cdot 4 \cdot 5 \cdot 6 = 360$

CSU Math Day 2005
Team Competition
Large School Final (Round 2)

1. A bag of Halloween candy has 7 pieces of chocolate and 9 pieces of fruit bar. What is the probability that 2 items selected at random are both chocolate?

   ANS: $\frac{\binom{7}{2}}{\binom{16}{2}} = \boxed{\frac{7}{40}}$

2. According to the Rational Root Theorem, what are all the possible rational roots of the polynomial $3x^3 - 4x^2 + 5x - 6$?

   ANS: $\boxed{\pm\frac{1}{3},\ \pm\frac{2}{3},\ \pm1,\ \pm2,\ \pm3,\ \pm6}$

3. The graph of the polynomial equation $y = x^3 - 2x^2 - x - 6$ crosses the x-axis only at $x = 3$. Factor the polynomial.

   ANS: $x^3 - 2x^2 - x - 6 = \boxed{(x-3)\left(x^2 + x + 2\right)}$

4. State the contrapositive of the statement, "If the pedals of a bicycle are rotated in a forward direction, then the back wheel turns in a forward direction."

   ANS: If the back wheel rolls backwards, then the pedals rotate backwards.

5. A circular pizza is diameter 14 inches is cut into 8 congruent slices. What is the perimeter of each slice?

   ANS: $14 + 2\pi 7/8 = \boxed{\frac{7}{4}\pi + 14}$

6. Morris Kline's most famous book is entitled A) Putting Boats in Kline Bottles, B) Why Johnny Can't Add, or C) The Man Who Counted.

   ANS: B) Why Johnny Can't Add

7. A vertical cylindrical tank has inside diameter of 6 feet and height of 8 feet. If 1 cubic foot of water weighs 62.4 pounds and 'a pint's a pound the world around', how many gallons of water (to the nearest 1000 gallons) will the tank hold?

   ANS: $3^2\pi 8\,\text{ft}^3 \times 62.5\,\text{lbf}/\,\text{ft}^3 \times 8\,\text{gal}/\,\text{lbf} = 113097.3355\,\text{gal} \approx \boxed{113\,000}$ gallons

8. In 1949, Selberg and Erdös wrote an elementary proof of which of the following theorems: A) Four Color Theorem, B) Fermat's Last Theorem, or C) Prime Number Theorem?
   ANS: C) Prime Number Theorem

Team Competition
Large School Final (Round 2)

9. If the 60 rearrangements of CONOCO are listed in alphabetic order, where does the word CONOCO appear in this list?

   ANS: $\boxed{13th}$ CCNOOO CCONOO CCOONO CCOOON CNCOOO CNOCOO CNOOCO CNOOOC COCNOO^9 COCONO^{10} COCOON^{11} CONCOO^{12} CONOCO^{13}

   $\frac{5!}{3!} = 20$ start with C

10. To the nearest minute, at what time between 1:30 p.m. and 2:00 p.m. are the minute hand and the hour hand pointing in opposite directions?

    ANS: $\boxed{1{:}38}$ p.m. $5 + \frac{m}{12} + 30 = m$, Solution is: $\frac{420}{11} = 38.18181818$

11. A computer sequentially computes integers by the following rule: If n is a square then multiply by 2; otherwise subtract 3. Starting at $n = 12$, what is the integer after 6 iterations?

    ANS: $12 \to 9 \to 18 \to 15 \to 12 \to 9 \to \boxed{18}$

12. The vertices of a quadrilateral are at the points $(0,0)$, $(3,1)$, $(4,0)$, and $(2,-4)$. What is the area of the quadrilateral?

    ANS: $Area = \frac{1}{2} \cdot 4 \cdot 1 + \frac{1}{2} \cdot 4 \cdot 3 = \boxed{8}$

13. A boat with a constant propeller speed goes 30 miles upstream in 5 hours and the same 30 miles back downstream in 3 hours. Find the velocity of the stream.
    ANS: $\boxed{1}$ mile per hour

    $(r - s)\,5 = 30, (r + s)\,3 = 30$, Solution is: $r = 8, s = \boxed{2}$ mph

14. The Simpsons own an SUV that gets 12 mi/ gal and a two-seater that gets 60 mi/ gal. They drive each vehicle 10000 miles per year. What is their average mileage?

    ANS: $\dfrac{20000}{\frac{10000}{12} + \frac{10000}{60}} = 20\,\text{mi/ gal}$

15. Edith used 99 digits to write page numbers on her paper for English class. How many pages was her paper?

ANS: $\boxed{54}$ since $9 + 2 \times 45$

Tiebreaker Question

How many edges does an icosahedron have?
ANS: $\frac{3 \times 20}{2} = \boxed{30}$

CSU Math Day 2005
Team Competition
Emergency Exam

1. Two 2-digit numbers are created using the digits 1, 2, 3, and 4 once each. The product of these numbers is calculated. What is the largest possible product?

   ANS: $41 \cdot 32 = \boxed{1312}$ $(42 \cdot 31 = 1302,\ 43 \cdot 21 = 903)$

2. What is the maximum number of pieces into which a circular pizza can be cut using 4 chops of a knife (with no intermediate rearrangements of the pieces)?

   ANS: $\boxed{11}$

3. What is the smallest whole number greater than 1 that is a perfect square, a perfect cube, and a perfect fourth power?
   ANS: $2^{12} = (2^6)^2 = (2^4)^3 = (2^3)^4 = \boxed{4096}$

4. The product of three consecutive odd integers is 2145. What is the sum of these three integers?

   ANS: $11 \cdot 13 \cdot 15 = 2145$ and $13 \times 3 = \boxed{39}$

5. Use the integers 1, 2, 3, and 4 each once to replace the variables a, b, c, and d. That is the maximum possible value for the expression $a + b \cdot c^d$?

   ANS: $1 + 2 \cdot 3^4 = \boxed{163}$ $1 + 2 \times 4^3 = 129$ $1 + 4 \times 2^3 = 33$

6. Two evenly-matched baseball teams, the Rockies and the Yankees, start a 7-game series. In how may different ways (RRRYYR, etc.) can the Rockies win the series four games to two?

   ANS: The Rockies win game 6 plus 3 of the first 5, so $\frac{5!}{3!2!} = \binom{5}{3} = \boxed{10}$

7. What is the largest integer that can be stored in a 8-bit computer word?

   ANS: $\boxed{255}$

8. A Spanish port was protected by large cannons, and each cannon had a pile of cannon balls nearby stacked neatly in the shape of a tetrahedron. If each bottom layer contained a total of 28 cannon balls, how many cannon balls were in each pile?

   ANS: $28 + 21 + 15 + 10 + 6 + 3 + 1 = \boxed{84}$

Team Competition
Emergency Exam

9. When this Hungarian mathematician died recently, the New York Times acknowledged his contributions to mathematics that included some 1500 published papers. Name this person.

   ANS: Paul $\boxed{\text{Erdös}}$

10. What is the square root of the cube root of 729?

    ANS: $\sqrt{\sqrt[3]{729}} = \boxed{3}$

11. What is the smallest positive integer that cannot be written as a sum of distinct digits?
    ANS: $\boxed{46}$ since $\sum_{n=1}^{9} n = 45$

12. What is the smallest positive integer exactly divisible by 2, 3, 4, 5, 6, and 7?

    ANS: $\text{lcm}\,(2, 3, 4, 5, 6, 7) = \boxed{420} = 2^2 \times 3 \times 5 \times 7$

13. The determinant of the matrix of coefficients of a system of two linear equations in two unknowns is 0. What is true about the graphs of the two equations?

    ANS: $\boxed{\text{The lines are parallel}}$

14. Sarah averages 50% on multiple choice exams. What is the probability that she gets at least 5 correct on a 9-question exam?

    ANS: $\boxed{\frac{1}{2}}$

15. What are the next two prime numbers greater than 50?

    ANS: $51 = 3 \times 17, \boxed{53}, 55 = 5 \times 11, 57 = 3 \times 19, \boxed{59}$

Tiebreaker Question

What is the prime factorization of 87?
ANS: $87 = \boxed{3 \times 29}$

CSU Math Day 2005
Team Competition
Emergency Exam (v2)

1. Let $y = mx + b$ be the image when the line $3x + 8y + 9 = 0$ is reflected across the x-axis. What is $m + b$?

   ANS: $3x + 8y + 9 = 0$, Solution is: $y = -\frac{3}{8}x - \frac{9}{8}$, reflected line is $y = \frac{3}{8}x + \frac{9}{8}$ and $\frac{3}{8} + \frac{9}{8} = \boxed{\frac{3}{2}}$

2. A basketball team has 12 players on the roster. How many different starting lineups are possible?

   ANS: $\binom{12}{5} = \boxed{792}$

3. Assume m and n are natural numbers such that $m > n$. If n is added to the sum of the first m natural numbers, the result is 2300. Find $m + n$.

   ANS: $m = 67$ and $n = 22$ so that $m + n = 67 + 22 = \boxed{89}$

4. What is the largest integer $\leq \sqrt{600}$?

   ANS: $\sqrt{600} = 24.49489743$, so $\boxed{24}$

5. What is the area of one face of a diamond-shaped kite whose two sticks are 20 and 50 inches long?

   ANS: $\boxed{500}$

6. How many three-digit numbers have a units digit larger than the tens digit?

   ANS: $45 \cdot 9 = \boxed{405}$

7. What is the radius of a sphere whose positive volume (in cubic meters) is the same as its surface area (in square meters)?
   ANS: $\boxed{3}$ meters $\frac{4}{3}\pi r^3 = 4\pi r^2$, Solution is: $3, 0$

8. Name the Scottish mathematician who invented logarithms.

   ANS: $\boxed{\text{Napier}}$

Team Competition
Emergency Exam (v2)

9. In which quadrant do the two lines $y = 26 + 2x$ and $y = 6 - 3x$ intersect?

   ANS: $\left\{ \begin{array}{c} y = 26 + 2x \\ y = 6 - 3x \end{array} \right\}$, Solution is : $(-4, 18)$ $\boxed{\text{second quadrant}}$

10. What are all the possible rational roots of $2x^3 - 5x^2 - x + 8 = 0$, as limited by the Rational Root Theorem?

    ANS: $\boxed{\pm\frac{1}{2}, \pm 1, \pm 2, \pm 4, \pm 8}$

11. How many positive integral factors does 72 have?

    ANS: $\boxed{12}$ $72 = 2^3 3^2$

12. An automobile travels at 90 kilometers per hour for 6 minutes, 15 km/ h for 2 minutes, and then 30 km/ h for 5 minutes. How far did it travel during the 13 minutes?

    ANS: $\frac{90}{10} + \frac{15}{30} + \frac{30}{12} = \boxed{12}$ km

13. How far apart are the two points at which the curves $y = x + 6$ and $y = x^2$ intersect?

    ANS: $\left\{ \begin{array}{c} y = x + 6 \\ y = x^2 \end{array} \right\}$, Solution is : $\{(-2, 4), (3, 9)\}$, distance $\boxed{5\sqrt{2}}$

14. Find the coordinate of the point on the number line that is one-third of the way from 1/2 to 7/8. Express your answer as a common fraction.

    ANS: $\boxed{\frac{5}{8}} = \frac{2}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{7}{8}$

15. What is the contrapositive of the statement, "If $x > 4$ then $x^2 > 16$"?

    ANS: $\boxed{\text{If } x^2 \leq 16\text{, then } x \leq 4.}$

Tiebreaker Question

How many faces does a tetrahedron have?
ANS: $\boxed{4}$

CSU Math Day 2005
Team Competition
Extra Problems

1. This 20th century American mathematician introduced game theory as a mathematical discipline, conceived the idea of a self-stored computer program, and worked on the Manhattan project that developed the atomic bomb. Name this person.

   ANS: John $\boxed{\text{von Neuman}}$

2. What is the area of a regular hexagon of edge 1?

   ANS: 6 triangles of base 1, height $\sqrt{3}/2 = 6 \times \left(\frac{1}{2}\right) \times 1 \times \frac{\sqrt{3}}{2} = \boxed{\frac{3}{2}\sqrt{3}}$

3. Who posed the famous 23 problems defining the future of mathematics in 1900?
   ANS: David $\boxed{\text{Hilbert}}$

4. A deck of 52 cards is thoroughly shuffled and the cards are turned over two at a time. How many pairs (two cards of the form $\boxed{\text{A}}\boxed{\text{A}}$ or $\boxed{5}\boxed{5}$) do you expect to see?

   ANS: $26\left(\frac{3}{51}\right) = \boxed{\frac{26}{17}}$

5. When an integer is divided by 15, the remainder is 7. Find the sum of the remainders when the same integer is divided by 3 and by 5.

   ANS: $\boxed{3} = (7 \bmod 3) + (7 \bmod 5)$

6. What is the prime factorization of $4! = 24$?

   ANS: $2^3 3$

7. Partition 25 into two parts such that the difference of their square roots is 1.

   ANS: $\boxed{9,\ 16}$

8. In how many ways can 4 math books, 3 stat books, and 2 physics books be arranged on a shelf, assuming the books must be in groups by their category? (ie: math with math, stat with stat, etc.)

   ANS: $\boxed{1728} = (2!)(3!)^2(4!)$

Team Competition
Extra Problems

9. A 3-dimensional cube has 8 vertices and 12 edges. How many vertices and how many edges does a 4-dimensional cube have?

   ANS: 16 vertices and 32 edges

10. Three line segments joining the midpoints of the sides of a triangle determine a smaller triangle whose sides lie inside the larger triangle. What is the ratio of the area of the larger triangle to the area of the smaller triangle?

    ANS: $\boxed{4:1}$

11. Find an integer between 100 and 1000 that is both a perfect square and a perfect cube.

    ANS: $\boxed{729} = 9^3 = 27^2$

12. Which is largest; π, 355/113, or 3.1416?

    ANS: $355/113 \approx 3.14159292$, $\pi \approx 3.141592654$, $\boxed{3.1416}$

13. Archie, Bob, and Claire are now 11, 17, and 19 years old, respectively. In how many years will they again have prime-numbered ages?

    ANS: $\boxed{\text{Twelve years from now}}$ they will have ages 23, 29, and 31.

14. The average of 3 numbers is 14. The number 12 is discarded from the collection of 3 numbers. What is the average of the remaining 2 numbers?

    ANS: $\boxed{15}$

15. If $i^2 = -1$, what is $(i-1)^3$?
    ANS: $2 + 2i$

Tiebreaker Question

Sometimes, always, or never: The sum of two irrational numbers is irrational.
ANS: $\boxed{\text{sometimes}}$ $\sqrt{2} + \sqrt{3}$ is irrational, $\sqrt{2} + \left(1 - \sqrt{2}\right) = 1$ is rational.