

Part I: Short answer

1. A rectangular box has the side lengths x , x and y . Suppose all box sides shrink by taking away $1/4$ of each side. By what factor does the box volume x^2y decrease?
2. A cube of side length 4 is inscribed in a sphere so that the corners of the cube touch the sphere. What is the volume of the sphere?
3. Write $\sqrt{\frac{4}{3}} - \sqrt{\frac{3}{4}}$ in the form $\frac{a}{b}$ where both a and b are either integers or roots of integers.
4. If $9^x = 243 = 3 \cdot 81$, then $x = ?$
5. In a group of cows and chickens, the number of legs is 14 more than twice the number of heads. What is the number of cows?

Part II: Show all work

6. Show that among any set of five integers, there are always three whose sum is divisible by three.
7. A tile-layer has a mixture of tiles that are of shapes 2×2 and 1×4 . She tiles a rectangular floor with them and finds that she has exactly one tile left over. One of the tiles on the floor is accidentally smashed, and sadly, it is not the the same shape as the left over one. Show that the tile-layer cannot rearrange the remaining tiles (including the one left-over) to retiling the floor.
8. In a group of 20 people each pair of friends exchanges holiday cards (assume that friendship is mutual). Among three people, however, there are always two who are not friends. Show that at most 200 cards are written.
9. An octagon that has 4 consecutive sides of length 2 and the four remaining sides of length 3 is inscribed in a circle. Determine the area of the octagon in the form $r + s\sqrt{t}$ with r , s and t rational numbers.
10. Let a and b be positive real numbers such that the cubic $x^3 - ax + b$ has three real roots, and let r be the root of smallest absolute value. Show that $\frac{b}{a} < r \leq \frac{3b}{2a}$.
11. Imagine placing solid spheres (possibly of different radii) around a point light source in such a way that:
 - (a) The light source is not inside any sphere,
 - (b) The spheres do not intersect,
 - (c) The spheres will block all light coming from the source.

What is the minimum number of spheres needed to achieve this? Justify your answer.