2002 PROBE

Part I: Short answer (4 versions)

1. Suppose a real number x satisfies [$x^4 - 2x^3 + 4x^2 - 8x + 16$, $x^4 + 2x^3 + 4x^2 + 8x + 16$, $x^4 - \frac{1}{2}x^3 + \frac{1}{4}x^2 - \frac{1}{8}x + \frac{1}{16}$, $x^4 + \frac{1}{2}x^3 + \frac{1}{4}x^2 + \frac{1}{8}x + \frac{1}{16}$]. What is x^5?

2. In how many different ways can you write [$900 = 2^2 \cdot 3^2 \cdot 5^2$. $756 = 2^2 \cdot 3^3 \cdot 7$, $1260 = 2^2 \cdot 3^2 \cdot 5 \cdot 7$, $840 = 2^3 \cdot 3 \cdot 5 \cdot 7$] as a product of three numbers [Question: do we allow the case when one of the numbers is 1? If not, what language to exclude it?].

3. Take a circle C of diameter [1, 2, 3, 4] tangent to a circle B of diameter [3, 1, 2, 1] in such a way that neither circle is contained in the other. Construct A of diameter [4, 3, 5, 5] containing both C and B. What fraction of the area of A is not conained in either B or C? [Better wording?]

4. Porbability?

5. Mystery?

Part II: Show all work

6. Suppose that a real number x can be expressed as

$$x = \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{3 + \cdots}}}}.$$

   What is x? [Need wording here...Maybe "Find a finite algebraic expression for x?" Other suggestions?]

7. Consider an arrangement n lines in the plane such that

   - No two of the lines are parallel.
   - No three of the lines pass through a single point.

   Such an arrangement divides the plane into regions. Prove that the number of these regions is independent of the arrangement, and find an expression for this number as a function of n.

8. Consider a long strip of paper. Fold it in half n times, by folding the left end over the right at each step. If you now unfold the strip, you will see a pattern of "up" and "down" creases. Describe this pattern for any n.

9. An equilateral triangle T_0 has sides of length 1. The midpoints of T_0 are joined by straight lines to form a new equilateral triangle T_1. Next, the midpoints of T_1 are joined by straight lines to form a new equilateral triangle T_2, and so on. After n steps we arrive at an equilateral triangle T_n. What is the area of T_n?

10. Show that there is an n such that 3^n (in base 10) contains 2002 consecutive 0 digits.

11. Let A, B, C be angles in a triangle. Show that

$$\tan(A) + \tan(B) + \tan(C) = \tan(A)\tan(B)\tan(C)$$