

Part I: Short Answer Give a short answer to the following questions in the space provided. You do not need to show work.

1. What is the angle (in degrees) formed by the hands of a clock at 2:15?

ANSWER: 22.5°

2. Eunice and Eugene bought identical boxes of stationary. Eunice used hers to write 1-sheet letters while Eugene used his to write 3-sheet letters. In this way, Eunice used all of her envelopes and had 50 sheets of paper left, while Eugene used all of his sheets and had 50 envelopes left. How many sheets of paper were in each box?

ANSWER: 150 sheets

3. What is the sum of the real solutions x of the equation $|x + 2| = 2|x - 2|$?

ANSWER: 20/3

4. What is the last digit of (the base 10 expression for) 3^{2007} ?

ANSWER: 7

5. The centers of the six faces of a $2 \times 2 \times 2$ inch cube are the vertices of a regular octahedron. What is the octahedron's volume?

ANSWER: 4/3 cubic inches

Part I: Short Answer Give a short answer to the following questions in the space provided. You do not need to show work.

1. What is the angle (in degrees) formed by the hands of a clock at 2:40?

ANSWER: 160°

2. Eunice and Eugene bought identical boxes of stationary. Eunice used hers to write 1-sheet letters while Eugene used his to write 3-sheet letters. In this way, Eunice used all of her envelopes and had 60 sheets of paper left, while Eugene used all of his sheets and had 60 envelopes left. How many sheets of paper were in each box?

ANSWER: 180 sheets

3. What is the sum of the real solutions x of the equation $|x + 1| = 2|x - 3|$?

ANSWER: 26/3

4. What is the last digit of (the base 10 expression for) 3^{2009} ?

ANSWER: 3

5. The centers of the six faces of a $1 \times 1 \times 1$ inch cube are the vertices of a regular octahedron. What is the octahedron's volume?

ANSWER: 1/6 cubic inches

Part I: Short Answer Give a short answer to the following questions in the space provided. You do not need to show work.

1. What is the angle (in degrees) formed by the hands of a clock at 2:20?

ANSWER: 50°

2. Eunice and Eugene bought identical boxes of stationary. Eunice used hers to write 1-sheet letters while Eugene used his to write 3-sheet letters. In this way, Eunice used all of her envelopes and had 40 sheets of paper left, while Eugene used all of his sheets and had 40 envelopes left. How many sheets of paper were in each box?

ANSWER: 120 sheets

3. What is the sum of the real solutions x of the equation $|x + 4| = 2|x - 1|$?

ANSWER: 16/3

4. What is the last digit of (the base 10 expression for) 3^{2008} ?

ANSWER: 1

5. The centers of the six faces of a $3 \times 3 \times 3$ inch cube are the vertices of a regular octahedron. What is the octahedron's volume?

ANSWER: 9/2 cubic inches

Part I: Short Answer Give a short answer to the following questions in the space provided. You do not need to show work.

1. What is the angle (in degrees) formed by the hands of a clock at 2:40?

ANSWER: 160°

2. Eunice and Eugene bought identical boxes of stationary. Eunice used hers to write 1-sheet letters while Eugene used his to write 3-sheet letters. In this way, Eunice used all of her envelopes and had 30 sheets of paper left, while Eugene used all of his sheets and had 30 envelopes left. How many sheets of paper were in each box?

ANSWER: 90 sheets

3. What is the sum of the real solutions x of the equation $|x + 3| = 2|x - 2|$?

ANSWER: 22/3

4. What is the last digit of (the base 10 expression for) 3^{2010} ?

ANSWER: 9

5. The centers of the six faces of a $4 \times 4 \times 4$ inch cube are the vertices of a regular octahedron. What is the octahedron's volume?

ANSWER: 32/3 cubic inches

Part II: Long Answer Answer the following questions as completely as possible. Show all work for partial credit.

6. Twenty-seven gold $1 \times 1 \times 1$ cubes are assembled into a $3 \times 3 \times 3$ cube and the surface of the big cube is painted green. After the paint dries, the big cube is disassembled and the 27 pieces are randomly scrambled before they are again assembled into a $3 \times 3 \times 3$ cube. What is the probability that the surface of the new big cube is again entirely green? (You do not need to simplify your answer!)

SOLUTION: The pattern of the paint breaks the little cubes into four groups: corners, edges, centers (of faces), and middle (meaning the very middle of the big cube.) Now, there are 24 ways to orient a cube, but if the paint pattern is to match, only some of those work. The data are summarized below:

Type	Number	Correct Orientations
Corner	8	3
Edge	12	2
Center	6	4
Middle	1	24

Thus, of the $24^{27}27!$ ways to reassemble the cube, only $(3^8!)(2^{12}12!)(4^6!)(24^11!)$ get the paint-job right, and the correct probability is

$$\frac{(3^8!)(2^{12}12!)(4^6!)(24^11!)}{24^{27}27!} = \frac{12!8!6!}{24^{18}27!} \approx 1.8 \times 10^{-37}.$$

7. A circular disk of radius r is inscribed in a square. Four smaller disk, each tangent to the big disk, are inscribed in the corners of the square. Four even smaller disks, tangent to the previous disks, are inscribed in the remaining corners. The process continues *ad infinitum*, adding four disks at each step. What fraction of the area of the square is covered by disks?

SOLUTION: The key observation is the following: if a circle of radius x is inscribed in a square and a circle of radius y is inscribed in one of the leftover corners, then the segment of length $x + y$ connecting the centers of the two circles is the hypotenuse of an isosceles right triangle with leg length $x - y$. It follows from the Pythagorean theorem that

$$(x + y)^2 = 2(x - y)^2.$$

Solving this quadratic equation and discarding the geometrically meaningless solution, obtain

$$y = x(3 - 2\sqrt{2}).$$

For simplicity, set

$$a = (3 - 2\sqrt{2})^2 = 9 - 12\sqrt{2} + 8 = 17 - 12\sqrt{2}.$$

Since the desired ratio is independent of r , we assume $r = 1$. Then the total area of all of the disks is

$$A = \pi(1 + 4a + 4a^2 + 4a^3 + \dots) = \pi(1 + 4a(1 + a + a^2 + \dots))$$

(where the factor of 4 reflects that there is a sequence of little circles in each corner.) Summing the geometric series, we obtain

$$A = \pi(1 + 4a/(1 - a))$$

Since the original square has area 4 (remember, $r = 1$), the desired ratio is

$$\frac{\text{Area of disks}}{\text{Area of square}} = \frac{A}{4} = \pi \frac{1 + 3a}{4(1 - a)} = \frac{\pi}{8}(3\sqrt{2} - 2) \approx 0.88$$

8. Five suspicious sailors are stranded on a remote island. They spend the day gathering a pile of coconuts. Exhausted, they postpone dividing it until the next morning. Being suspicious, however, each decides to take a share during the night.

The first sailor divides the pile into five equal portions plus one extra coconut, which he gives to a monkey. He takes and hides one portion and leaves the rest in a single pile. The second sailor later does the same; again the monkey receives one leftover coconut. The third, fourth and fifth sailor also do this; each time a remainder of one goes to the monkey. In the morning, they split the remaining coconuts into five equal piles, and each gets his "share". (Each knows some were taken, but none complains, since each is guilty!)

What is the smallest possible number of coconuts in the original pile?

Let m be the final share each sailor receives in the morning. For $i = 1, 2, 3, 4, 5$, let n_i be the number of coconuts in the pile before the i th sailor does his sneaky business (so n_1 is the initial pile size that we are trying to find) and let n_6 be the number in the final pile before it gets divided in the morning.

The equations relating these quantities are:

$$4(n_i - 1) = 5n_{i+1}$$

for $i = 1, 2, 3, 4$ and 5 , and

$$5m = n_6.$$

Now working backwards from the last equation and simplifying

$$n_1 = \frac{1}{4^4}(5^6(m/4) + 5^4 + 5^3 \cdot 4 + 5^2 \cdot 4^2 + 5 \cdot 4^3) + 1 = \frac{1}{4^4}(5^6(m/4) + 1845) + 1.$$

Thus, we are looking for the smallest positive integer m such that $(5^6(m/4) + 1845)$ is divisible by $4^4 = 256$.

Working now modulo $4^4 = 256$, we compute that $1845 \equiv 53 \pmod{256}$ while $5^6 \equiv 9 \pmod{256}$. Applying the Euclidean algorithm, compute

$$1 = 57 \cdot 9 - 256 \cdot 2$$

so 57 is a multiplicative inverse of 9 modulo 256 .

Therefore, $(5^6(m/4) + 1845)$ is divisible by $4^4 = 256$ if and only if

$$m/4 \equiv -53 \cdot 57 \equiv 51 \pmod{256}.$$

Therefore, we conclude that the smallest possibility for each final share is $m = 4 \cdot 51 = 204$, which leads to an initial pile of

$$n_1 = \frac{1}{4^4}(5^6 \cdot 51 + 1845) + 1 = 3121.$$

9. $*$ is a real-valued binary operation on the non-zero reals which satisfies

(a) $a * a = 1$

(b) $a * (b * c) = (a * b)c$ (the usual product of $(a * b)$ with c)

for all a, b , and c . Solve the equation

$$x * 36 = 216$$

for x .

SOLUTION: Multiply both sides of the equation by 36 and apply the properties:

$$6^5 = (216)(36) = (x * 36)36 = x * (36 * 36) = x * 1.$$

Now if y positive real number, $1 = 6^y * 6^y$ by the first property. So

$$6^5 = x * (6^y * 6^y) = (x * 6^y)6^y$$

so that

$$x * 6^y = 6^{5-y}$$

for all y . Now setting $y = \log_6 x$,

$$1 = x * x = 6^5/x,$$

whence

$$x = 6^5.$$

NOTE: In fact, there is nothing special about the number 6 in this problem and one can easily show that $x * y = x/y$. But the proof does require the use of logs, and hence, that $*$ really be defined for all positive real inputs.

10. If x and y are positive real numbers such that

$$x^3 + y^3 + (x + y)^3 + 30xy = 2000,$$

show that $x + y = 10$.

SOLUTION:

If x and y are solutions, then

$$\begin{aligned} 0 &= x^3 + y^3 + (x + y)^3 + 30xy - 2000 \\ &= (x + y)^3 - 3(x^2y + xy^2) + (x + y)^3 + 30xy - 2000 \\ &= 2((x + y)^3 - 1000) - 3xy(x + y - 10) \\ &= 2(x + y - 10)((x + y)^2 + 10(x + y) + 100) - 3xy(x + y - 10) \\ &= (x + y - 10)(2(x^2 + 2xy + y^2 + 10x + 10y + 100) - 3xy) \\ &= (x + y - 10)(2x^2 + xy + 2y^2 + 20x + 20y + 200). \end{aligned}$$

Since both x and y are positive, the second factor is never 0, and therefore it follows that $(x + y - 10) = 0$ or

$$x + y = 10.$$

11. A biologist models the spread of a population as follows. First, she divides the area of study into a square, $n \times n$ grid of cells. Then, she notes which cells are initially populated. Now each new week, any previously populated cell remains populated, and any unpopulated cell adjacent (vertically and/or horizontally, but *not* diagonally) to at least two previously populated cells also becomes populated. After many weeks, the biologist finds that all n^2 cells are populated. Prove that at least n squares were initially populated.

SOLUTION:

Consider an unpopulated cell C with exactly two adjacent populated cells D and E . Regardless of whether D and E abut C on opposite or adjacent sides, *the perimeter of the populated region does not change when C becomes populated*. Try it!

On the other hand, if an unpopulated cell C has three or four adjacent populated cells, then the perimeter of the populated region must decrease when C becomes populated.

The net effect is that as more and more cells become populated, *the total perimeter of the populated region can only remain constant or decrease*. Since the perimeter of the entire grid is $4n$, the initial populated zone must have had perimeter at least $4n$, which requires a minimum of n initially populated cells.

NOTE: As a matter of fact, starting with the cells of one diagonal populated leads, after $(n - 1)$ weeks, to a fully populated grid. Of course, there are other interesting initial configurations of n populated cells that do the trick too.