CHAPTER 7

The Equivalence Theorem

7.1. Perturbing an \( m \)-equivalence

In proving the equivalence theorem, starting from one free and ergodic \( G \)-action, with two \( m \)-equivalent arrangements we will need to construct a second one which will be, in a sense we now make precise, a perturbation of the first. The aim of this will be to show that a certain open set of \( m \)-joinings is in fact dense under appropriate hypotheses. This density will be shown by demonstrating how to “perturb” any given \( m \)-joining into the given open set. In this section we will develop the technical facts we will need about such perturbations.

Recall that when we omit the subscript \( x \) on an expression \( f_x^{\alpha, \beta} \)) or \( h_x^{\alpha, \beta} \) we are regarding them as maps from \( X \) to \( \mathcal{R} \) or \( \mathcal{G} \). Suppose \((X, \mathcal{F}, \mu, T^\omega)\) is a free and ergodic \( G \)-action, \( \{\phi_i\} \) is a sequence of elements in the full group of \( T^\omega \) for which

\[
(f_1^{\alpha_1, \phi_1} \otimes f_2^{\alpha_2, \phi_2} \otimes \ldots \otimes f_n^{\alpha_n, \phi_n} \otimes f_j^{\alpha_j, \phi_j})^* \mu \in \mathcal{M}^m.
\]

The following discussion will be relative to this fixed action and sequence of full group elements.

Under these conditions we know that \( \alpha \phi_i \to \beta \) where \( \alpha \sim \beta \). We describe now what we mean by an \( I_0, J, \delta \)-perturbation of this \( G \)-action. It is perhaps more correct to call it an \( m \)-perturbation, but as we will assume \( m \) is a fixed size from here on, we omit this.

Throughout the rest of this chapter, it will be convenient to have a metric for the weak*-topologies on various spaces of measures. Let \( D \) represent such a metric in all cases.

**Definition 7.1.1.** Fix values \( I_0 \) and \( J \) and consider the measure

\[
\hat{\mu} = (f_1^{\alpha_1, \phi_1} \otimes f_2^{\alpha_2, \phi_2} \otimes \ldots \otimes f_n^{\alpha_n, \phi_n} \otimes f_j^{\alpha_j, \phi_j})^* \mu \in \mathcal{M}_e(\mathcal{R}^{k+1}).
\]

Suppose we have a second free and ergodic \( G \)-action \((X_1, \mathcal{F}_1, \mu_1, T_1^{\omega_1})\) and \( I_0 + 1 \) elements in its full group \( \phi_1^1, \phi_1^2, \ldots, \phi_1^{k+1} \), and a further sequence of full group elements \( \{\psi_i\} \) satisfying:

i) \( D(\hat{\mu}, (f_1^{\alpha_1, \phi_1^1} \otimes \ldots \otimes f_1^{\alpha_1, \phi_{k+1}^1})^* \mu_1) < \delta \) and
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ii) \( (f^{\alpha_1 \phi_1} \otimes f^{\alpha_1 \phi_2} \otimes \ldots)^* \mu_1 \in \mathcal{M}^m \text{ and} \)

iii) \( m((f^{\alpha_1 \phi_1+1} \psi_1 \otimes f^{\alpha_1 \phi_2+1} \psi_2 \otimes \ldots)^* \mu_1) < \delta. \)

We refer to such an action and collection of full-group elements as an \( I_0, J, \delta \)-perturbation of the original \( G \)-action and sequence in the full group.

Notice that in such an \( I_0, J, \delta \)-perturbation the sequence of rearrangements \( \alpha_1 \phi_1+1 \psi_i \) will converge in \( i \) to an arrangement we will call \( \beta_1 \), and \( \alpha_1 \) and \( \beta_1 \) will be \( m \)-equivalent. What interests us is the precise sequence of full group elements

\[
\phi_1', \phi_2', \ldots, \phi_{I_0+1}' \psi_1, \phi_{I_0+1}' + 1 \psi_2, \ldots \]

taking \( \alpha_1 \) to \( \beta_1 \). We will write this sequence as \( \{\phi''_i\} \).

Notice that if we drop to a subsequence of the \( \psi_i \) in an \( I_0, J, \delta \)-perturbation, we will still have an \( I_0, J, \delta \)-perturbation. What we will show in the rest of this section is that given a \( G \)-action \( (X, \mathcal{F}, \mu, T^\alpha) \) and sequence of full group elements \( \{\phi_i\} \) with

\[
(f^{\alpha_1 \phi_1} \otimes f^{\alpha_2 \phi_2} \otimes \ldots)^* \mu \in \mathcal{M}^m,
\]

then for any \( I_0 \), if \( J \) is large enough and \( \delta \) small enough, any \( I_0, J, \delta \)-perturbation will still put

\[
(f^{\alpha_1 \phi_1''} \otimes f^{\alpha_2 \phi_2''} \otimes \ldots)^* \mu_1 \in \mathcal{M}^m.
\]

To begin to see why, notice the following calculation:

For all \( i \leq I_0 + 1 \),

\[
m_{\alpha_1}(\phi''_i, \phi''_j) = \begin{cases} 
  m_{\alpha_1}(\phi', \phi'_j), & j \leq I_0 + 1 \\
  m_{\alpha_1}(\phi'_i, \phi'_{I_0+1} - I_0 - 1), & j > I_0 + 1 
\end{cases}
\]

\[
\leq \begin{cases} 
  m_{\alpha_1}(\phi', \phi'_j), & j \leq I_0 + 1 \\
  m_{\alpha_1}(\phi', \phi'_{I_0+1}) + m_{\alpha_1}(\psi_j - I_0 - 1), & j > I_0 + 1 
\end{cases}
\]

\[
\leq \begin{cases} 
  m_{\alpha_1}(\phi', \phi'_j), & j \leq I_0 + 1 \\
  m_{\alpha_1}(\phi', \phi'_{I_0+1}) + \delta, & \text{for all } j > I_0 + 1 \text{ sufficiently large.}
\end{cases}
\]

Lemma 7.1.2. For any free and ergodic \( G \)-action \( (X, \mathcal{F}, \mu, T^\alpha) \) and full-group elements \( \phi_i \) with

\[
(f^{\alpha_1 \phi_1} \otimes f^{\alpha_2 \phi_2} \otimes \ldots)^* \mu_1 \in \mathcal{M}^m
\]
and value $I_0$, there exists $\delta > 0$ such that for all $J$ sufficiently large, any $I_0, J, \delta$-perturbation (dropping to a subsequence of the $\{\psi_i\}$ if necessary) will satisfy:

$$\limsup_{j \to \infty} m_{\alpha_1}(\phi_i^j, \phi_j^u) < 1/i$$

and

$$\limsup_{j \to \infty} m_{\beta_1}(\phi_i^{j-1}, \phi_j^{u-1}) < 1/i.$$ 

**Proof.** As $(f^{\alpha, \phi_1} \otimes \ldots)^* \mu \in \mathcal{M}$ we have, for all $i$,

$$\limsup_{j \to \infty} m_{\alpha}(\phi_i, \phi_j) < 1/i$$ 

and

$$\limsup_{j \to \infty} m_{\beta}(\phi_i^{j-1}, \phi_j^{u-1}) < 1/i.$$ 

Hence, for all $i$, both of these lim sup’s are in fact limits converging to $m_{\alpha}(\phi_i, \langle \phi_j \rangle_\alpha)$ and $m_{\beta}(\phi_i^{j-1}, \langle \phi_j^{u-1} \rangle_\beta)$ both of which must be $< 1/i$. Hence there is an $\varepsilon_1 > 0$ so that for all $i \leq I_0$ we have

$$m_{\alpha}(\phi_i, \langle \phi_j \rangle_\alpha) < 1/i - \varepsilon_1.$$ 

Thus for all $J$ sufficiently large, and $i = 1, \ldots, I_0$

$$m_{\alpha}(\phi_i, \phi_j) < 1/i - \varepsilon_1.$$ 

By Axiom 3, there exists $\delta > 0$ such that if

$$D(\hat{\mu}, (f^{\alpha_1, \phi_i} \otimes \ldots \otimes f^{\alpha_n, \phi_{I_0+1}})^* \mu_1) < \delta$$

then all of the finite list of inequalities

$$m_{\alpha_1}(\phi_i^j, \phi_j^{I_0+1}) < 1/i - \varepsilon_1, \quad i = 1, \ldots, I_0 + 1$$

will still hold.

As long as $\delta < \varepsilon_1/2$, the calculation made preceding this lemma implies

$$m_{\alpha_1}(\phi_i^j, \phi_j^{I_0+1} \psi_j) \leq m_{\alpha_1}(\phi_i^j, \phi_j^{I_0+1}) + m_{\alpha_1 \phi_j}^{I_0+1}(\text{id}, \psi_j) < 1/i - \varepsilon_1/2$$

and for $i = 1, \ldots, I_0$

$$\limsup_{j \to \infty} m_{\alpha_1}(\phi_i^j, \phi_j^u) < 1/i.$$ 

For $i = I_0 + 1$,

$$\limsup_{j \to \infty} m_{\alpha_1}(\phi_i^j, \phi_j^u) = m_{\alpha_1 \phi_j}^{I_0+1}(\text{id}, \psi_j - I_0 - 1) < \delta.$$
Making sure that $\delta < 1/(I_0 + 1)$ we obtain this term. For $i > I_0 + 1$, as

$$\limsup_{j \to \infty} m_{\alpha_1}(\phi_i^\alpha, \phi_j^\alpha) = \limsup_{j \to \infty} m_{\alpha_1} \phi_{n+1}^\alpha (\psi_{i-1}, \psi_j) = m_{\alpha_1} \phi_{n+1}^\alpha (\psi_{i-1}, \alpha_1)$$

by dropping to a subsequence of the $\psi_i$ we can ensure that this value is $< 1/i$.

As we assume that the sequence $\phi_i^\alpha$ achieves an $m$-equivalence between $\alpha_1$ and $\beta_1$, the other set of strict inequalities now follows automatically from Lemma 6.4.5. \[\square\]

We now want to obtain the same fact for pointwise convergence, that a small enough perturbation will not leave $\mathcal{M}_{**}$. The first step is to notice that from Axiom 2 we get the following:

**Lemma 7.1.3.** For all $\delta > 0$ there exists $\delta_1 > 0$ and if $(X_1, \mathcal{F}_1, \mu_1, T_{\alpha_1})$ is a free and ergodic $G$-action, with $\phi_{n+1}^\alpha$ and $\{\psi_i\}$ in its full group with $\langle \psi_i \rangle \alpha_1 \in \hat{\mathcal{M}}(\alpha_1)$ and

$$m_{\alpha_1} \phi_{n+1}^\alpha (\langle \psi_i \rangle \alpha_1, \text{id}) < \delta_1,$$

then for a subsequence of the $\psi_i$ we have both

$$\int \sup_i (d(h^{\alpha_1} \phi_{n+1} \phi_i, \text{id})) d\mu_1 < \delta$$

and

$$\int \sup_i (d(h^{\beta_1} \phi_i^{-1}, \text{id})) < \delta.$$

**Proof.** Axiom 2 tells us that if $\delta_1$ is sufficiently small than for all $i$ sufficiently large,

$$\int d(h^{\alpha_1} \phi_{n+1} \phi_i \psi_i, \text{id}) d\mu_1 < \delta/4.$$

We can drop to a subsequence of the $\psi_i$ with

$$d(h^{\alpha_1} \phi_{n+1} \phi_i \psi_i, h^{\alpha_1} \phi_{n+1} \beta_i) \to 0$$

for $\mu_1$-a.e. $x$. Hence by the Lebesgue dominated convergence theorem,

$$\int d(h^{\alpha_1} \phi_{n+1} \beta_i, \text{id}) \leq \delta/4.$$

Thus by omitting sufficiently many initial terms $\psi_i$ we can obtain
\[ \sup_i (d(h_x^\alpha \phi_{i+1}^{\psi'}, h_x^\alpha \phi_{i+1}^{\psi_k})) \leq \delta/4 \]

for a subset of \( X \) of measure \( > 1 - \delta/4 \).

As \( d \leq 1 \), restricting to this tail of the \( \psi_i \) sequence we obtain

\[ \int \sup_i (d(h_x^\alpha \phi_{i+1}^{\psi}, \text{id})) \, d\mu_1 \leq 3\delta/4 < \delta. \]

As the sequence \( \{\psi_i\} \) realizes the \( m \)-equivalence between \( \alpha_1 \phi_{i+1}^{\psi} \) and \( \beta_1 \), we also have

\[ m_{\beta_1}((\psi_i^{-1})_{\beta_1}, \text{id}) < \delta_1 \]

and the second inequality of the lemma follows symmetrically. \( \square \)

**Lemma 7.1.4.** Suppose \((X, F, \mu, T^\alpha)\) is a free and ergodic \( G \)-action, such that \( \{\phi_i\} \) is in its full group with

\[ (f^{\alpha \phi_1} \otimes f^{\alpha \phi_2} \otimes \ldots)^* \mu \in \mathcal{M}^m. \]

Given any \( \varepsilon > 0 \) and \( I_0 \), there exists \( \delta > 0 \) and \( J > I_0 \) so that if \((X_1, F_1, \mu_1, T_1^\alpha)\), \( \phi_i' \), \ldots, \( \phi_{J+1}^{i+1} \) and \( \{\psi_i\} \) form an \( I_0, J, \delta \)-perturbation of the first system, then we can conclude

a) for all \( i \leq I_0 + 1 \)

\[ \int R_{i, I_0 + 1}(h^\alpha \phi_{i+1}^{\psi}, h^\alpha \phi_{i+1}^{\psi}) \, d\mu_1 \]

\[ \leq \int R_{i, I_0 + 1}(h^\alpha \phi_{i}, h^\alpha \phi_{i}) \, d\mu + \varepsilon, \]

b) \( \int \sup_i (d(h^\alpha \phi_{i+1}^{\psi}, h^\alpha \phi_{i+1}^{\psi}) \, d\mu_1 \leq \varepsilon, \)

c) for all \( i \leq I_0 + 1 \),

\[ \int R_{i, I_0 + 1}(h^{\alpha_1 \phi_{i+1}^{\psi}}, h^{\alpha_1 \phi_{i+1}^{\psi}}) \, d\mu_1 \]

\[ \leq \int R_{i, I_0 + 1}(h^{\alpha_1 \phi_{i}^{\psi}}, h^{\alpha_1 \phi_{i}^{\psi}}) \, d\mu + \varepsilon \]

and

d) \( \int \sup_i (d(h^{\alpha_1 \phi_{i+1}^{\psi}}, h^{\alpha_1 \phi_{i+1}^{\psi}}) \, d\mu_1 < \varepsilon. \)
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Proof. To begin, for any \( \delta_1 > 0 \) and \( I_0 \), there is a \( \delta > 0 \) such that for all \( J \) sufficiently large, perhaps dropping to a subsequence of the \( \psi_i \), we will have:

i) \( D((f^{\alpha, \phi_1} \otimes \ldots \otimes f^{\alpha, \phi_0} \otimes h^{\alpha, \beta}*\mu, (f^{\alpha, \phi_1} \otimes \ldots \otimes f^{\alpha, \phi_0} \otimes h^{\alpha, \phi_0} + 1)*\mu_1) < \delta_1, \)

\[ \int \sup_i(d(h^{\alpha, \phi_0 + 1, \phi_0 + 1, \psi_i, \mu})) d\mu_1 < \delta_1 \]

ii) \( \int \sup_i(d(h^{\alpha, \phi_0 + 1, \phi_0 + 1, \psi_i, \mu})) d\mu_1 < \delta_1 \) and

iii) \( \int \sup_i(d(h^{\beta, \psi_i, \mu_1})) d\mu_1 < \delta_1 \).

To obtain i) just notice that \( h^{\alpha, \phi, \beta}_x \rightarrow h^{\alpha, \beta}_x \) and the map \( H \) taking \( \mathcal{R} \) to \( \mathcal{G} \) is continuous. The second two statements follow directly from the Lemma 7.1.3. The proofs of a)–d) will follow, except for one small step, from i)–iii) with \( \delta_1 \) small enough and \( J \) large enough.

One can calculate that for any two \( G \)-arrangements \( \alpha \) and \( \beta \) and full group element \( \phi \) that

\[ f^{\beta, \phi^{-1}}_x(g) = h^{\alpha, \beta}_x(h^{\alpha, \alpha}_x(h^{\beta, \phi}_x(g)f^{\alpha, \phi}_x(id)))g^{-1}. \]

Represent a point in \( \mathcal{R}^k \times \mathcal{G} \) as \((f'_1, f'_2, \ldots, f'_k, \ell')\), and define

\[ F(f', \ell')(g) = \ell'(H(f')^{-1}(\ell'^{-1}(g)f'(id)))g^{-1}, \]

and notice that the maps

\[ (f'_1, f'_2, \ldots, f'_k, \ell') \rightarrow H(F(f'_i, \ell')), \quad i = 1, \ldots, I_0 \]

are all continuous.

It follows that

\[ H(F(f^{\alpha, \phi_1}_x, h^{\alpha, \phi_0 + 1}_x)) = h^{\alpha, \phi_0 + 1}_x(h^{\alpha, \phi_0 + 1}_x + 1\phi_0 + 1). \]

Partition \( \mathcal{R}^k \times \mathcal{G} \) into a countable collection of clopen sets according to the values

\[ H(F(f'_1, \ell'))(g_j), H(f'_j)(g_j), \ell'(g_j) \text{ and } \]

\[ H(F(f'_1, \ell'))^{-1}(g_j), H(f'_j)(g_j), \ell'(g_j), \quad j = 1, \ldots, [\ln(8/e)] + 1. \]

Label the partition elements \( C_1, C_2, \ldots \). Notice that for any two points \((f'_1, \ldots, f'_k, \ell')\) and \((f''_1, \ldots, f''_k, \ell'')\) which belong to the same \( C_k \) we will have
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$$d(\ell', \ell'') < \varepsilon/8$$
$$d(H(F(f_i', \ell')), H(F(f_i'', \ell''))) < \varepsilon/8 \text{ and}$$
$$d(H(f_i'), H(f_i'')) < \varepsilon/8, \quad i = 1, \ldots, I_0.$$  

Choose a finite list $C_1, \ldots, C_K$ with

$$\mu\left(\bigcup_{k=1}^K (f_{\alpha, \phi_1} \otimes \cdots \otimes f_{\alpha, \phi_{I_0}} \otimes h^{\alpha, \beta})^{-1}\left(C_k\right)\right) > 1 - \varepsilon/8.$$  

As the $C_k$ are clopen, we can choose $\delta_1$ so small that (i) implies

$$\left( f_{\alpha_1, \phi_1'} \otimes \cdots \otimes f_{\alpha_1, \phi_{I_0}} \otimes h^{\alpha_1, \alpha_1, \phi_{I_0} + 1}\right)^n \mu_1\left(\bigcup_{k=1}^K C_k\right) > 1 - \varepsilon/4.$$  

Select an index $N > \lceil \ln(8/\varepsilon) \rceil + 1$ so large that for all $g_j, \ j = 1, \ldots, \lceil \ln(8/\varepsilon) \rceil + 1$ and $(f_1', \ldots, f_{I_0}', \ell') \in \bigcup_k C_k$ all of the values

$$H(f_i')(g_j), H(f_i')^{-1}(g_j), H(F(f_i', \ell'))(g_j),$$

$$H(F(f_i', \ell'))^{-1}(g_j), \ell'(g_j), \text{ and } \ell'^{-1}(g_j)$$

are indexed as some $g_n, \ n \leq N$. (Remember, all of these values are constant on each set $C_k$.)

Set $\varepsilon = 1/N$ and be sure

$$\delta_1 < \frac{\varepsilon \varepsilon}{8(I_0 + 3)}$$

and (ii) now implies

$$\int d\left(h^{\alpha_1, \phi_{I_0} + 1, \beta_1}, \text{id}\right) d\mu_1 \leq \delta_1.$$  

Define a set $A$ by
A = \{ x_1 : d(h_{x_1}^{\alpha_1 \phi'_{l_0+1} \alpha_1 \phi'_{l_0+1} \psi}, \text{id}) \leq e \text{ and } \\
d(h_{\phi'_j(x_1)}^{\alpha_1 \phi'_{l_0+1} \alpha_1 \phi'_{l_0+1} \psi}, \text{id}) \leq e \\
\text{for } j = 1, \ldots, I_0 \text{ and all } i, \text{ hence } \\
d(h_{x_1}^{\alpha_1 \phi'_{l_0+1} \beta_1}, \text{id}) \leq e \text{ and } \\
d(h_{\phi'_j(x_1)}^{\alpha_1 \phi'_{l_0+1} \beta_1}, \text{id}) \leq e \\
\text{for } j = 1, \ldots, I_0, \text{ and we further require } \\
d(h_{\phi'_{l_0+1}(x_1)}^{\alpha_1 \phi'_{l_0+1} \psi}, \text{id}) \leq e \text{ and } \\
d(h_{\phi'_{l_0+1}(x_1)}^{\beta_1 \beta_1 \psi}, \text{id}) \leq e \text{ for all } i\}. 

As there are \( I_0 + 3 \) inequalities to satisfy to lie in \( A \), each of which holds on a set of measure at least \( \delta_1/e \) (remember that \( \mu_1 \) is preserved by all the \( \phi'_j \)) we conclude

\[ \mu(A) > 1 - \varepsilon/8. \]

Let

\[ A' = A \cap (f^{\alpha_1 \phi'_1} \otimes \cdots \otimes f^{\alpha_1 \phi'_0} \otimes f^{\alpha_1 \phi'_0 + 1})^{-1}(\bigcup_{k=1}^{K} C_k), \]

and \( \mu_1(A') > 1 - 3\varepsilon/8. \)

The four inequalities come in two pairs (a),(b) and (c),(d). The latter two are more difficult so we focus on their proofs. We begin with (c). For all \( x_1 \in A' \), \( i = 1, \ldots, I_0 \) and \( j = 1, \ldots, [\ln(8/\varepsilon)] + 1 \) we will have

\[ H(F(f_{x_1}^{\alpha_1 \phi'_1}, h_{x_1}^{\alpha_1 \beta_1}))(g_j) = h_{x_1}^{\beta_1 \alpha_1 \phi'_{l_0+1}}(g_j) \]

\[ = h_{x_1}^{\alpha_1 \phi'_{l_0+1} \beta_1 \alpha_1 \phi'_{l_0+1}}(g_j) = h_{x_1}^{\alpha_1 \phi'_{l_0+1} \alpha_1 \phi'_{l_0+1} \beta_1 \alpha_1 \phi'_{l_0+1} \psi}(g_j) \]

as both the pre- and post- functions in this composition act on \( g_j \) and its image as the identity. Taking the inverses of all the bijections in this calculation, we find the same identity holds there. That is to say:
\[ h^{\beta_{j+1}, \beta_1} \left( g_j \right) = h^{\alpha_{j+1}, \beta_1, \alpha_{j+1}, \beta_1} \left( h_{x_1}^{\alpha_{j+1}, \beta_1, \alpha_{j+1}, \beta_1} \left( g_j \right) \right) \]

Examining the \((I_0 + 1)\)st term and \(j = 1, \ldots, \lfloor \ln(8/\varepsilon) \rfloor + 1,\)

\[ h^{\beta_{j+1}, \beta_1} \left( g_j \right) = h^{\alpha_{j+1}, \beta_1, \alpha_{j+1}, \beta_1} \left( h_{x_1}^{\alpha_{j+1}, \beta_1, \alpha_{j+1}, \beta_1} \left( g_j \right) \right) \]

and again the same identity holds at these \(g_j\) for the inverse maps. Hence for \(x_i \in A'\) and all \(i = 1, \ldots, I_0 + 1,\)

\[ d\left( h^{\beta_{j+1}, \beta_1, \alpha_{j+1}, \beta_1, \alpha_{j+1}, \beta_1} \left( h_{x_1}^{\alpha_{j+1}, \beta_1, \alpha_{j+1}, \beta_1} \left( g_j \right) \right) \right) \leq \varepsilon / 8. \]

Calculating c),

\[
\int R_{i, l_0+1} (h^{\beta_{j+1}, \beta_1, \alpha_{j+1}, \beta_1, \alpha_{j+1}, \beta_1}) \mu_1
\]

\[
\leq \int R_{i, l_0+1} (h^{\alpha_{j+1}, \beta_1, \alpha_{j+1}, \beta_1, \alpha_{j+1}, \beta_1}) \mu_1 + 5\varepsilon / 8
\]

\[
\leq \int R_{i, l_0+1} (h^{\alpha_{j+1}, \beta_1, \alpha_{j+1}, \beta_1, \alpha_{j+1}, \beta_1}) \mu_1 + 5\varepsilon / 8
\]

The map

\[ (f^1_{l_0+1}, f^2_{l_0+1}, \ldots, f^l_{l_0+1}) \rightarrow R_{i, l_0} (H(F(f^1_{l_0+1}), H(F(f^2_{l_0+1}), \ldots, H(F(f^l_{l_0+1}, H(F(f^l_{l_0+1}, H(f_{l_0+1}) \right)) \]

is continuous from \(R^{l_0+1} \rightarrow [0, 1]\) and so if \(\delta\) is small enough we will obtain from i) of the Definition 7.1.1 of an \(I_0, J, \delta\)-perturbation that this calculation is

\[
\leq \int R_{i, l_0+1} (h^{\alpha_{j+1}, \beta_1, \alpha_{j+1}, \beta_1, \alpha_{j+1}, \beta_1}) \mu_1 + 3\varepsilon / 4.
\]

As \(h^{\beta_1, \alpha_{j+1}} \rightarrow \text{id}\) for all \(x,\) if \(J\) now is sufficiently large, this is

\[
\leq \int R_{i, l_0} (h^{\beta_1, \beta_1, \alpha_{j+1}, \beta_1, \alpha_{j+1}, \beta_1}, h^{\beta_1, \beta_1, \alpha_{j+1}, \beta_1, \alpha_{j+1}, \beta_1}) \mu_1 + \varepsilon
\]

which is c).

To obtain a) we can omit the first part of the argument for c) and just notice that for \(i = 1, \ldots, I_0 + 1,\) the maps

\[ (f^1_{l_0+1}, \ldots, f^l_{l_0+1}) \rightarrow R_{i, l_0+1} (H(f^1_{l_0+1}), \ldots, H(f^l_{l_0+1})) \]
taking \( \mathcal{R}_{\delta + 1} \rightarrow [0, 1] \) are continuous and so from i) of the definition of an \( I_0, J, \delta \)-perturbation, if \( \delta \) is small enough,

\[
\int R_{i, h+1}(h^{\alpha_1 \alpha_1 \phi_1, \ldots, h^{\alpha_1 \alpha_1 \phi_{h+1}}}) \, d\mu_1 
\leq \int R_{i, h+1}(h^{\alpha_1 \alpha_1 \phi_1, \ldots, h^{\alpha_1 \alpha_1 \phi_{h+1}}, h^{\alpha_1 \alpha_1 \phi_{h+1}}}) + \varepsilon \over 2
\]

and now if \( J \) is chosen sufficiently large, this will be

\[
\int R_{i, h}(h^{\alpha_1 \alpha_1 \phi_1, \ldots, h^{\alpha_1 \alpha_1 \phi_{h+1}}, h^{\alpha_1 \alpha_1 \phi_{h+1}}}) \, d\mu + \varepsilon
\]

which is a).

To demonstrate d) notice that for \( j = 1, \ldots, [\ln(8/\varepsilon)] + 1 \), and \( x_1 \in A' \) that

\[
h_{x_1}^{\beta_1 \beta_1 \phi_i^{-1}}(g_j) = h_{x_1}^{\alpha_1 \phi_i^{-1}}(g_j) h_{x_1}^{\alpha_1 \phi_i^{-1}}(x_1) h_{x_1}^{\alpha_1 \phi_i^{-1}}(g_j) = h_{x_1}^{\alpha_1 \phi_i^{-1}}(g_j)
\]

again as the pre- and post- composing functions act as the identity on \( g_j \) and its image. Also as before the same identity holds for the inverse bijections at \( g_j \). We also have

\[
h_{x_1}^{\beta_1 \beta_1 \psi_i^{-1} \phi_i^{-1}}(g_j) = h_{x_1}^{\alpha_1 \phi_i^{-1} \psi_i^{-1}}(g_j) = h_{x_1}^{\alpha_1 \phi_i^{-1} \psi_i^{-1}}(x_1) h_{x_1}^{\alpha_1 \phi_i^{-1} \psi_i^{-1}}(g_j).
\]

For all \( x_1 \) we know that \( h_{x_1}^{\beta_1 \beta_1 \psi_i^{-1} \phi_i^{-1}}(g_j) = g_j \) once \( i \) is large enough. Drop to a sufficiently distant tail of the sequence \( \psi_i \) so that on a set \( A'' \) with \( \mu_1(A'') > 1 - \varepsilon / 8 \), for \( x_1 \in A'' \) we have

\[
h_{x_1}^{\beta_1 \beta_1 \psi_i^{-1} \phi_i^{-1}}(g_j) = g_j \text{ for all } j = 1, \ldots, N.
\]

It follows that for \( x_1 \in A' \cap A'' \) we have

\[
h_{x_1}^{\beta_1 \beta_1 \psi_i^{-1} \phi_i^{-1}}(g_j) = h_{x_1}^{\alpha_1 \phi_i^{-1} \psi_i^{-1}}(g_j),
\]

and the same identity holds on the inverses of these bijections at \( g_j \) (note: it is this last remark that required us to use \( N \) in the definition of \( A'' \) and not \( [\ln(8/\varepsilon)] + 1 \).)
This implies for all \( x_1 \in A' \cap A'' \), for all \( i \) and for \( 1 \leq j \leq \lfloor \ln(8/\varepsilon) \rfloor + 1 \) that
\[
h_{x_i}^{\beta_1; \beta_i \phi^{-1}_i} (g_j) = h_{x_i}^{\beta_1; \beta_i \psi_i^{-1}} (g_j)
\]
and the same identity holds for the inverses of these bijections at \( g_j \).
Hence
\[
\int \sup_i d(h_{x_i}^{\beta_1; \beta_i \phi^{-1}_i} , h_{x_i}^{\beta_1; \beta_i \psi_i^{-1}}) \, d\mu_1 
\leq \int \sup_i d(h_{x_i}^{\beta_1; \beta_i \phi^{-1}_i} , h_{x_i}^{\beta_1; \beta_i \psi_i^{-1}}) \, d\mu_1 + \mu_1(A' \cup A'') 
\leq \varepsilon + \frac{\varepsilon}{8} < \varepsilon .
\]

For (b) we follow similar but easier lines. Notice that for \( j = 1, \ldots, \lfloor \ln(8/\varepsilon) \rfloor + 1 \) and \( x_1 \in A' \) that
\[
h_{x_1}^{\alpha_1; \alpha_i \phi^{-1}_i + \psi_i} (g_j) = h_{x_1}^{\alpha_1; \alpha_i \phi'_{i+1}} \psi_i \, h_{x_1}^{\alpha_1; \alpha_i \phi'_{i+1}} (g_j) = h_{x_1}^{\alpha_1; \alpha_i \phi'_{i+1}} (g_j)
\]
and the same identity holds at \( g_j \) for the inverse bijections. Now integrate just as for (d) to obtain (b).

**Theorem 7.1.5.** For \((X, \mathcal{F}, \mu, T^\alpha)\) a free and ergodic \( G \)-action and full group elements \( \phi_i \) with
\[
(f^{\alpha, \phi_1} \otimes f^{\alpha, \phi_2} \otimes \ldots)^* \mu \in \mathcal{M}^m
\]
and \( I_0 \), there is a \( \delta > 0 \) such that for all \( J \) sufficiently large, any \( I_0, J, \delta \)-perturbation
\[
(X_1, \mathcal{F}_1, \mu_1, T^{\alpha_1}_{1}, \phi'_{1}, \ldots, \phi'_{I_0+1}, \} \psi_i, 
\]
(by dropping to a subsequence of the \( \psi_i \)) will satisfy
\[
(f^{\alpha_1, \phi''_1} \otimes f^{\alpha_1, \phi''_2} \otimes \ldots)^* \mu_1 \in \mathcal{M}^m
\]
as well.

**Proof.** Having already proven in Lemma 7.1.2 that if this measure is in \( \mathcal{M}_\ast \) it will be in \( \mathcal{M}^m \) all we need to show is
\[
a) \int R_i(h_{x_i}^{\alpha_1; \alpha_i \phi''_i} , h_{x_i}^{\alpha_1; \alpha_i \phi''_{i+1}} , \ldots) \, d\mu_1 < (2^{i+2})^{-1} \text{ and}
\]
\[
b) \int R_i(h_{x_i}^{\beta_1; \beta_i \phi^{-1}_i} , h_{x_i}^{\beta_1; \beta_i \phi^{-1}_{i+1}} , \ldots) \, d\mu_1 < (2^{i+2})^{-1} .
\]
As \((f^{\alpha_1, \phi_1} \otimes f^{\alpha_1, \phi_1} \otimes \ldots)^* \mu_1 \in \mathcal{M}^m\) we already know that there must exist an \(\varepsilon_1 > 0\) so that for all \(i = 1, \ldots, I_0\),

\[
\int R_i(h^{\alpha, \alpha \phi_i}, \ldots) \, d\mu < (2^{i+2}i)^{-1} - \varepsilon_1 \quad \text{and} \\
\int R_i(h^{\beta, \beta \phi_i^{-1}}, \ldots) \, d\mu < (2^{i+2}i)^{-1} - \varepsilon_1,
\]

To demonstrate a) for \(i = 1, \ldots, I_0\), notice

\[
\int R_i(h^{\alpha_1, \alpha \phi_i''}, \ldots) \, d\mu_1 \\
\leq \int R_i(h^{\alpha_1, \alpha \phi_i}, \ldots, h^{\alpha_1, \alpha \phi_{I_0}}, h^{\alpha, \beta}) \, d\mu_1 \\
+ \int \sup_j d(h^{\alpha_1, \alpha \phi_i'_{I_0+1}, h^{\alpha_1, \alpha \phi_{I_0+1}}} \, d\mu_1.
\]

Applying Lemma 7.1.4 with \(\varepsilon'' = \varepsilon_1/3\), make sure \(\delta\) is at most the \(\delta''\) obtained there, and for all \(J\) sufficiently large we obtain from a) and b) of Lemma 7.1.4 that this calculation is

\[
\leq \int R_i(h^{\alpha_1, \alpha \phi_i}, \ldots, h^{\alpha_1, \alpha \phi_{I_0}}, h^{\alpha, \beta}) \, d\mu + 2\varepsilon_1/3 \\
\leq \int R_i(h^{\alpha, \alpha \phi_i}, h^{\alpha, \phi_{i+1}}, \ldots) \, d\mu
\]

(as \(h^{\alpha, \alpha \phi_i} \to h^{\alpha, \beta}\))

\[
\leq (2^{i+1}i)^{-1} - \varepsilon_1/3 < (2^{i+2}i)^{-1}.
\]

For \(i \geq I_0 + 2\),

\[
\int R_i(h^{\alpha_1, \alpha \phi_i''}, \ldots) \, d\mu_1 \\
= \int R_i(h^{\alpha_1, \alpha \phi_i'_{I_0+1}, \psi_i - I_0 - 1}, \ldots) \, d\mu_1
\]

and as \(\alpha_1 \phi_{I_0+1, \psi_i - I_0 - 1} \to \beta_1\), by dropping to a subsequence of the \(\psi_i\) we can ensure this is

\[
\leq \int \sup_j (2d(h^{\alpha_1, \alpha \phi_i'_{I_0+1}, \psi_i - I_0 - 1}, h^{\alpha_1, \beta_1})) \, d\mu_1 \\
< (2^{i+2}i)^{-1}.
\]

This completes a).
Obtaining b) follows parallel lines using c) and d) of Lemma 7.1.4. For \( i = 1, \ldots, I_0 + 1 \),

\[
\int R_i(h_{\beta_1, \phi_{i+1}^{-1}}, \ldots) \, d\mu_1 \\
\leq \int R_{i, I_0 + 1}(h_{\beta_1, \phi_{i+1}^{-1}}, \ldots, h_{\beta_1, \phi_{i+1}^{-1}}) \, d\mu_1 \\
+ \sup_i d(h_{\beta_1, \phi_{i+1}^{-1}}, h_{\beta_1, \phi_{i+1}^{-1}}) \, d\mu_1 \\
\leq \int R_i(h_{\beta_1, \phi_{i+1}^{-1}}, \ldots) \, d\mu + 2\varepsilon_1/3
\]

from c) and d) of Lemma 7.1.4 and that \( \beta \phi_{i+1}^{-1} \rightarrow \alpha \). But this is

\[
\leq (2^{i+1})^{-1} - \varepsilon_1/3 < (2^{i+2})^{-1}.
\]

For \( i \geq I_0 + 2 \) we calculate

\[
\int R_i(h_{\beta_1, \phi_{i+1}^{-1}}, \ldots) \, d\mu_1 \\
= \int R_i(h_{\beta_1, \phi_{i+1}^{-1}}, \ldots) \, d\mu_1
\]

and since \( \beta_1 \psi_{i-1}^{-1} \phi_{i+1}^{-1} \rightarrow \alpha_1 \), by dropping to a subsequence of the \( \psi_i \), just as for the previous case, we can ensure this is

\[
< (2^{i+2})^{-1}.
\]

\( \square \)

7.2. The \( \overline{m} \)-distance and \( m \)-finitely determined processes

**Definition 7.2.1.** Suppose \((Z, \mathcal{F}, \mu, T^\alpha)\) is a free and ergodic \( G \)-action and \( P : Z \rightarrow \Sigma \) is a finite labeled partition (that is to say, \( \Sigma \) is a finite labeling set). We refer to \((Z, \mathcal{F}, \mu, T^\alpha, P)\), as a \( \Sigma \)-valued process. We will usually abbreviate this by just the pair \((T^\alpha, P)\) as long as their is no confusion. We consider two \( \Sigma \)-valued processes to be identical if they give rise to the same \( \sigma \)-invariant measure on \( \Sigma^G \) via the usual map of a point in \( Z \) to its name \( \{P(T^\alpha_\beta(z))\}_{\beta \in G} \in \Sigma^G \).

If it is not necessary, we will also suppress the \( \Sigma \) and just refer to \((T^\alpha, P)\) as a process. This is in keeping with the usual vocabulary of the isomorphism theory of Ornstein and Weiss.
Definition 7.2.2. Given two $\Sigma$-processes $(T_{1}^{\alpha_{1}}, P_{1})$ and $(T_{2}^{\alpha_{2}}, P_{2})$, we define the $\overline{m}$-distance between them by

$$
\overline{m}(T_{1}^{\alpha_{1}}, P_{1}; T_{2}^{\alpha_{2}}, P_{2}) = \inf_{\hat{\mu} \in J_{m}(T_{1}^{\alpha_{1}}, T_{2}^{\alpha_{2}})} \left( \hat{\mu}(\{(z_{1}, z_{2}) : P_{1}(z_{1, id}) \neq P_{2}(z_{2, id})\}) + m(\hat{\mu}) \right).
$$

That is to say, we consider all $m$-joinings of the two actions, and among them look for the one which simultaneously matches the two labeled partitions on the two processes as closely as possible, and is as small an $m$-joining as possible.

For $\hat{\mu}$ an $m$-joining of two processes we abbreviate

$$
\hat{\mu}(\{(z_{1}, z_{2}) : P_{1}(z_{1, id}) \neq P_{2}(z_{2, id})\}) \text{ by } \hat{\mu}(P_{1} \triangle P_{2}).
$$

We refer to the evaluation $\hat{\mu}(P_{1} \triangle P_{2}) + m(\hat{\mu})$ as the $\overline{m}$-evaluation at $\hat{\mu}$.

Lemma 7.2.3. $\overline{m}$ is a metric on the space of all $\Sigma$-valued processes. Furthermore, for any $\varepsilon > 0$, the set of $m$-joinings $\hat{\mu}$ with

$$
\hat{\mu}(P_{1} \triangle P_{2}) + m(\hat{\mu}) < \overline{m}(T_{1}^{\alpha_{1}}, P_{1}; T_{2}^{\alpha_{2}}, P_{2}) + \varepsilon
$$

is open.

Proof. That $\overline{m}$ is symmetric follows from the fact that $\overline{q}$ interchanges the roles of the two processes in an $m$-joining but does not alter the calculation of the infimum.

The existence of the diagonal joining certainly tells us that the $\overline{m}$-distance between a process and itself is zero. For the other direction of this, suppose $\overline{m}(T_{1}^{\alpha_{1}}, P_{1}; T_{2}^{\alpha_{2}}, P_{2}) = 0$. Let $\hat{\mu}_{i}$ be a sequence of $m$-joinings of the two actions with the $\overline{m}$-evaluation converging to zero. Consider the projections of these measures to their $\Sigma^{G} \times \Sigma^{G}$-names alone, dropping all other coordinates. Certainly then, the measures must be converging to a measure supported on the diagonal. On the first coordinate it must be $p_{1}^{\ast}(\mu_{1})$, the projection of $\mu_{1}$ onto the space of $\Sigma$-names. As $m(\hat{\mu}_{i}) \to 0$, by Axiom 2 we must also have that the projections of the $\hat{\mu}_{i}$ to measures on $\mathcal{G}$ are converging to a point mass on the identity bijection. That is to say, the $\hat{\mu}_{i}$ converge on $\Sigma^{G} \times \Sigma^{G} \times \mathcal{G}$ to a joining (not just an orbit-joining) of the two processes. As it is supported on the diagonal, the two processes must be identical. For the triangle inequality, notice that if we have two $m$-joinings, $\hat{\mu}_{1}$ of
7.2. THE $\tilde{m}$-DISTANCE AND $m$-FINITELY DETERMINED PROCESSES

$(T_1^0, P_1)$ and $(T_2^{02}, P_2)$ and $\hat{\mu}_2$, of $(T_1^0, P_1)$ with $(T_3^{03}, P_3)$, then we can construct the relatively independent coupling of $\hat{\mu}_1$ and $\hat{\mu}_2$ over their common $(T_1^0, P_1)$-factor. An ergodic component of this $(\hat{\mu}_3)$ will be an ergodic and free action with three $m$-equivalent arrangements, (versions of $\alpha_1, \alpha_2$, and $\alpha_3$), and three $\Sigma$-valued partitions (versions of $P_1, P_2$, and $P_3$). In this fixed system, $m_{\hat{\mu}_3, \alpha_3}$ is a metric on its $m$-equivalence class of arrangements, and the calculation $\hat{\mu}_3(P_1, P_2)$ is a metric on $\Sigma$-valued partitions.

Projecting $\hat{\mu}_3$ to just its $Z_2 \times Z_3$-coordinates and partitions, $\hat{\mu}_3$ can be mapped directly to an $m$-joining of the two processes $(T_2^{02}, P_2)$ and $(T_3^{03}, P_3)$. The $m$-evaluation for this $m$-joining can be pulled back and evaluated on $\hat{\mu}_3$ where it will be maximized by the sum of the two $\tilde{m}$-calculations for the original two $m$-joinings $\hat{\mu}_1$ and $\hat{\mu}_2$.

That the set of $m$-joinings that get within $\varepsilon$ of the $m$-distance is open follows from an observation. Knowing the space of $m$-joinings is a conjugacy invariant means we can assume here that the maps $P_1$ and $P_2$ are continuous, (i.e. the partitions are into clopen sets) as this is simply a different choice of model. This means the $\tilde{m}$-evaluation is an upper semi-continuous function of the $m$-joining $\hat{\mu}$.

We now define the notion of an $m$-finitely determined process. To readers familiar with the isomorphism theory this is lifted directly from the corresponding notion there. Hence it will be automatic that for $m$ an entropy preserving size, the Bernoulli processes are $m$-finitely determined, giving a basic class of examples. In this context it is more natural to speak of $m$-finitely determined processes of zero $m$-entropy and of positive $m$-entropy. The Bernoulli processes provide examples of $m$-finitely determined processes of positive $m$-entropy. One must examine case by case whether $m$-finitely determined processes of zero $m$-entropy exist. The simplest example is, of course, that no free and ergodic finitely determined actions of zero entropy exist.

Having defined the $m$-finitely determined property, we will first show that it is an $m$-equivalence invariant. We will show more, that in fact any process that sits as a factor of an $m$-finitely determined process is again $m$-finitely determined.

Hence one can speak of an $m$-finitely determined $G$-action as one for which every partition is $m$-finitely determined.

Finally we will show that among the $m$-finitely determined actions, $m$-entropy is a complete invariant of $m$-equivalence, i.e. that any two $m$-finitely determined actions of the same $m$-entropy are in fact $m$-equivalent. We do this by showing that if $(X, \mathcal{F}, \mu, T^n)$ is $m$-finitely determined, and $(X_1, \mathcal{F}_1, \mu_1, T_1^n)$ is any other free and ergodic $G$-action
with $h_m(T_1^{\alpha_1}) \geq h_m(T_1^\alpha)$, then in the space of $m$-joinings $J_m(T_1^\alpha, T_1^{\alpha_1})$, those $\hat{\mu}$ for which the full group elements $\{\phi_i\}$ and the first coordinate algebra $\mathcal{F}_1$ are $\hat{\mu}$-a.s. $\mathcal{F}_1$-measurable, form a dense $G_\delta$ subset. If the two actions happened to both be $m$-finitely determined, and of the same $m$-entropy, then simply intersecting the two residual subsets, we see that $m$-equivalences between them not only exist, but form a dense $G_\delta$ subset of their space of $m$-joinings.

Before stating the definition of $m$-finitely determined, following our earlier convention, fix a metric $D$ giving the weak*-topology on the space of Borel measures on a sequence space $\Sigma^G$ where $\Sigma$ is some finite labeling space. Also remember, for $P : X \to \Sigma$, we let $p(x) = \{P(T_1^{\alpha}(x))\}_{\alpha \in G}$ be the $T_1^\alpha, P$-name of the point. Again as a reminder, we define a pseudometric

$$
\|((T, P), (T_1, P_1))\|_* = D(p^*(\hat{\mu}), p_1^*(\hat{\mu}_1)).
$$

**Definition 7.2.4.** We say a $\Sigma$-valued process $(T_1^\alpha, P)$ is **$m$-finitely determined** (abbreviated $m$-f.d.) if for any $\varepsilon > 0$ there is a $\delta > 0$ so that if $(T_1^{\alpha}, P_1)$ is any other $\Sigma$-valued process satisfying:

1. $\|((T, P), (T_1^{\alpha}, P_1))\|_* < \delta$ and
2. $h_m(T_1^{\alpha_1}, P_1) \geq h_m(T_1^{\alpha}, P) - \delta$,

then

3. $\overline{m}(T_1^{\alpha_1}, P_1; T_1^{\alpha}, P) < \varepsilon$.

A few comments are appropriate. To begin, condition (2) is a bit disingenuous in that if $m$ is entropy preserving, then it should read

$$
h(T_1^{\alpha_1}, P_1) \geq h(T_1^{\alpha}, P) - \delta
$$

and if $m$ is entropy free, then (2) is no condition at all.

We have mentioned earlier that among positive $m$-entropy processes (that is to say, positive entropy processes and entropy preserving sizes $m$), the Bernoulli processes give examples of $m$-f.d. processes. This is simply because for the f.d. processes of Ornstein and Weiss (1) and (2) imply $\delta$-closeness, which is to say, the existence of a joining $\hat{\mu}$ with

$$
\hat{\mu}(P \triangle P_1) < \varepsilon.
$$

Such a joining of course extends via a point mass on $\{\text{id}\}$ to an $m$-joining with $m(\hat{\mu}) = 0$. 

Although this definition is well-suited to the completion of a general equivalence theorem it suffers from being perhaps difficult to verify for any particular process. For 3+ sizes we can use a more easily verified condition. In its definition we use the notion of an **ergodic lift** of some $G$-action $T$ on a space standard space $(X, \mathcal{F}, \mu)$. By this we mean an ergodic $G$-action $\tilde{T}$ on some $(\hat{X}, \hat{\mathcal{F}}, \hat{\mu})$ which factors my some map $\pi$ onto $T$. Notice that the full group of $T$ then lifts as a subgroup of the full group of $\tilde{T}$. Further if $T = T^\alpha$ then any other arrangement $\beta$ of the orbit of $T$ lifts as well. Both of these observations follow from the fact that all functions on $X$, in particular $f_{\alpha, \phi}$, $q^{\alpha, \phi}$ and $h_{\alpha, \beta}$ lift to $\hat{X}$ through the projection of $\hat{x}$ to $\pi(\hat{x})$. The same of course applies to say any partition $P$ of $X$ can be regarded as a partition of $\hat{X}$.

**Definition 7.2.5.** For $T^\alpha$ a free free and ergodic $G$ action we say the $\Sigma$ valued process $(T^\alpha, P)$ is **weakly m-finitely determined** (weakly m-f.d.) if for each $\varepsilon > 0$ there is a $\delta > 0$ such that for any other free and ergodic $G$ action $T_0^{\alpha_0}$ and $\Sigma$ valued partition $P_0$ with

1. $\|(T^\alpha, P) , (T_0^{\alpha_0}, P_0)\|_* < \delta$ and

2. $h_m(T_0^{\alpha_0}, P_0) \geq h_m(T^\alpha, P) - \delta$

and each $\delta_1 > 0$ there is an ergodic lift $T_1^{\alpha_1}$ of $T_0^{\alpha_0}$, a $\Sigma$-valued partition $P_1$ of $X_1$ and a $\phi_1$ in the full group of $T_1^{\alpha_1}$ with

a. $\mu_1(P_0 \Delta P_1) < \varepsilon$ and

b. $m(\alpha_1, \phi_1) < \varepsilon$,

for which

1'. $\|(T^\alpha, P), (T_1^{\alpha_1, \phi_1}, P_1)\|_* < \delta_1$, and

2'. $h_m(T_1^{\alpha_1}, P_1) \geq h_m(T^\alpha, P) - \delta_1$.

Notice that this condition becomes particularly simple for an entropy free size where both conditions 2. and 2'. are vacuous.

**Theorem 7.2.6.** Suppose $T^\alpha$ is an ergodic and free action of $G$ and $P$ is a $\Sigma$-valued partition ($\Sigma$ a finite set).

1. For each size $m$ if $(T^\alpha, P)$ is m-finitely determined then it is weakly m-finitely determined.
2. For each $3^+$ size $n$ if $(T^\alpha, P)$ is weakly $m$-finitely determined then it is $m$-finitely determined.

**Proof.** For part 1, suppose $(T^\alpha, P)$ is $m$-finitely determined. If $\delta$ is small enough and $(T_0^\alpha, P_0)$ satisfies 1. and 2. of $m$-f.d. then we obtain

$$m(T_0^\alpha, P_0; T^\alpha, P) < \varepsilon$$

and there is a $\hat{\mu} \in J_m(T_0^\alpha, T^\alpha)$ with

$$\hat{\mu}(\{(z_1, z_2) : P_1(z_{1,\delta}) \neq P_2(z_{2,\delta}))\} + m(\hat{\mu}) < \varepsilon.$$ 

The action $\sigma^\alpha$ on this $m$-joining is an ergodic lift of $T_0^\alpha$ and

$$m(\alpha, \beta) < \varepsilon$$

which is to say

$$\lim_{i \to \infty} m(\alpha, \alpha \phi_i) < \varepsilon.$$ 

As the sequence $\alpha \phi_i$ is converging in $m_\hat{\mu}$ to $\beta$ we have

$$(\sigma^\alpha \phi_i, P_{2}(z_{2,\delta})) \to (\sigma^\beta, P_2(z_{2,\delta}))$$

in both distribution and $m$-entropy. Hence for this lift of $T_0^\alpha$ we can select $\phi = \phi_i$ for some $i$ and $P_1 = P_2(z_{2,\delta})$ and obtain a., b., 1', and 2'.

To verify 2. notice that the weakly $m$-f.d. condition is set to be used inductively moving from conditions 1. and 2. to 1' and 2', with a. and b. measuring the size of successive perturbations. Thus if $(T^\alpha, P)$ is weakly $m$-f.d. and $\varepsilon > 0$, then for any $(T_0^\alpha, P_0)$ satisfying 1. and 2. of the definition one can obtain a succession of ergodic lifts, partitions $P_i$ and full group elements $\phi_i$. To simplify notation we can assume all these ergodic lifts sit inside one maximal ergodic lift we call $T_0^\alpha$. From such an inductive application of the definition we obtain for an initial step:

$$m(\hat{\alpha}_0, \phi_1) < \varepsilon$$

and

$$\hat{\mu}(P_0 \triangle P_1) < \varepsilon$$

but after this first step, for $i \geq 1$

$$m(\hat{\alpha}_i \phi_i, \phi_i^{-1} \phi_{i+1}) < \varepsilon_i$$

and

$$\hat{\mu}(P_i \triangle P_{i+1}) < \varepsilon_i$$

where the values $\varepsilon_i$ are at our disposal to choose.

Hence we can force

$$\hat{\alpha}_i \overset{m}{\longrightarrow} \beta$$
with \( m(\hat{\alpha}, \hat{\beta}) < \varepsilon \) and

\[ P_i \longrightarrow \hat{P} \]

with \( \hat{\mu}(P_0 \Delta \hat{P}) < \varepsilon \).

As \( (\hat{T}_{0}^{\hat{\alpha}, \hat{\beta}}, P_i) \longrightarrow (T^\alpha, P) \) in distributions we will have

\( (\hat{T}_{0}^{\hat{\alpha}, \hat{\beta}}, \hat{P}) \equiv (T^\alpha, P). \)

If \( m \) is a \( 3^+ \) size then Theorem 2.2.7 ensures \( \hat{\beta} \hat{\phi}_i^{-1} \longrightarrow m \hat{\alpha}_0 \) and that \( \hat{\alpha}_0, \hat{\beta} \) and the sequence \( \phi_i \) form an \( m \)-joining in the weak sense of \( T_{0}^{\hat{\alpha}_0} \) and \( T^\alpha \) allowing us to conclude that

\[ m(T_{0}^{\hat{\alpha}_0}, R_0; T^\alpha, P) < \varepsilon. \]

\( \square \)

Our next goal is to see that the \( m \)-f.d. property is actually a property of a sub-\( \sigma \)-algebra, not just a partition. More precisely, for any process \( (T^\alpha, P) \), let \( \mathcal{H}(T^\alpha, P) \) be the \( \sigma \)-algebra

\[ \bigvee_{g \in G} T_g^\alpha(P), \]

the smallest \( T^\alpha \)-invariant \( \sigma \)-algebra relative to which \( P \) is measurable. What we want to demonstrate is the following.

**Theorem 7.2.7.** If \( (T^\alpha, P) \) is \( m \)-f.d., and \( Q : X \to \Sigma_Q \) is any finite \( \mathcal{H}(T^\alpha, P) \)-measurable partition, then \( (T^\alpha, Q) \) is \( m \)-f.d.

We will argue this assuming \( m \) is entropy-preserving. The entropy-free case follows precisely the same lines, but without the need for any entropy estimates. The result will follow from the next two lemmas.

**Lemma 7.2.8.** Suppose \( (T^\alpha, P) \) is \( m \)-f.d. and \( Q : X \to \Sigma_Q \) is another finite partition with

\[ \mathcal{H}(T^\alpha, P) = \mathcal{H}(T^\alpha, Q). \]

Then \( (T^\alpha, Q) \) is \( m \)-f.d.

**Proof.** We prove this for \( m \) an entropy preserving size. Fix \( \varepsilon > 0 \) and choose \( K_0 \subseteq G \), a finite subset, so that there is a “coding”

\[ \alpha_0 : \Sigma_{P_0} \to \Sigma_Q \]

satisfying

\[ \mu(\{ x : \alpha_0(\{ P(T_g^\alpha(x)) \}_{g \in K_0}) \neq Q(x) \}) < \varepsilon /10. \]
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Choose \( \varepsilon_0 < \frac{\varepsilon}{10 \# K_0} \) small enough that on any orbit space for any pair of \( m \)-equivalent \( G \)-arrangements \( \alpha_1 \) and \( \alpha_2 \) with \( m(\alpha_1, \alpha_2) < \varepsilon_0 \), by Axiom 2 we will have

\[
h_x^{\alpha_1, \alpha_2} |_{K_0} = \text{id}
\]

for all but at most \( \varepsilon/10 \) in measure of the points \( x \).

Use \( \varepsilon_0 \) in the definition of \( m \)-f.d. for the process \((T^\alpha, P)\) to obtain a value \( \delta_0 \). Choose \( \varepsilon < \delta_0/10 \) with

\[
\varepsilon \log(\# \Sigma_Q) + H(\varepsilon) < \delta_0/10
\]

and

\[
\varepsilon < \frac{\varepsilon_0}{10 \# K_0}.
\]

Now once more choose finite codes \( c_1 \) and \( c_2 \) (much more accurate than \( c_0 \))

\[
c_1 : (\Sigma_Q)^K \to \Sigma_P \text{ and } c_2 : (\Sigma_P)^K \to \Sigma_Q,
\]

\( K \) a finite subset of \( G \), satisfying

a) \( \mu(\{x : c_1(\{Q(T^\alpha_g(x))\}_{g \in K}) = P(x)\}) > 1 - \varepsilon, \)

and letting \( \hat{P}(x) = c_1(\{Q(T^\alpha_g(x))\}_{g \in K}), \)

b) \( \mu(\{x : c_2(\{\hat{P}(T^\alpha_g(x))\}_{g \in K}) = c_2(\{P(T^\alpha_g(x))\}_{g \in K}) = Q(x)\})

> 1 - \varepsilon. \)

For any \( \Sigma_Q \)-valued process \((T^\alpha_1, Q_1)\) let

\[
\hat{P}_1(x_1) = c_1(\{Q_1(T^\alpha_1_{g,1}(x_1))\}_{g \in K}).
\]

Choose \( \delta > 0 \) so that if

1. \( \|(T, Q), (T_1, Q_1)\|_* < \delta \) then

1'.

\[
\|(T^\alpha, P), (T^\alpha_1, \hat{P}_1)\|_* \leq \|(T^\alpha, \hat{P}), (T^\alpha, P)\|_*
\]

\[
+\|(T^\alpha, \hat{P}), (T^\alpha_1, \hat{P}_1)\|_* < \delta_0,
\]

and furthermore both

\[
\mu_1(\{x_1 : c_0(\{\hat{P}_1(T^\alpha_{g,1}(x_1))\}_{g \in K}) = Q_1(x_1)\}) > 1 - \varepsilon/5
\]
and

\[ \mu_1(\{x_1 : c_2(\{\hat{P}_1(T_{i,g}^{\alpha_1}(x_1))\}_{g \in K}) = Q_1(x_1)\}) > 1 - 2\varepsilon. \]

Hence

\[ h(T_1^{\alpha_1}, \hat{P}_1) \geq h(T_1^{\alpha_1}, Q_1) - 2\varepsilon \log(\#\Sigma_q) - H(2\varepsilon) \]

\[ > h(T_1^{\alpha_1}, Q_1) - \delta/10. \]

Thus if

2. \( h(T_1^{\alpha_1}, Q_1) > h(T^{\alpha}, Q) - \delta = h(T^{\alpha}, P) - \delta \) then

2'. \( h(T_1^{\alpha_1}, \hat{P}_1) > h(T^{\alpha}, P) - \delta_0 \), and hence

3'. \( \overline{m}(T_1^{\alpha_1}, \hat{P}_1; T^{\alpha}, P) < \varepsilon_0 \).

Let \( \hat{\mu} \) be an ergodic joining of \( T^{\alpha} \) and \( T_1^{\alpha_1} \) for which the \( \overline{m} \)-evaluation is less than \( \varepsilon_0 \), that is to say, for which

\[ \hat{\mu}(\hat{P}_1 \Delta P) < \varepsilon_0 \text{ and } m(\hat{\mu}) < \varepsilon_0. \]

We conclude that

\[ \hat{\mu}(Q_1 \Delta Q) < \hat{\mu}(\{(p_1(x_1), p(x), \{f_i\}) : \text{ either} \]

\[ \quad \text{i. } \ell(H(f_i))(g) \neq g \text{ for some } g \in K_0, \]

\[ \quad \text{ii. } \hat{P}_1(T_{i,g}^{\alpha_1}(x_1)) \neq P(T_g^{\alpha}(x)) \text{ for some } g \in K_0, \]

\[ \quad \text{iii. } c_0(\{\hat{P}_1(T_{i,g}^{\alpha_1}(x_1))\}_{g \in K_0}) \neq Q_1(x_1) \text{ or,} \]

\[ \quad \text{iv. } c_0(\{P(T_g^{\alpha}(x))\}_{g \in K_0}) \neq Q(x) \}

\[ \leq \varepsilon/10 + \#K_0\varepsilon_0 + \varepsilon/5 + \varepsilon/10 < \varepsilon/2. \]

Since \( m(\hat{\mu}) < \varepsilon_0 < \varepsilon/10 \), this implies

\[ \overline{m}(T_1^{\alpha_1}, Q_1; T^{\alpha}, Q) < \varepsilon, \]
and hence \((T^\alpha, Q)\) is m-f.d. \(\square\)

**Lemma 7.2.9.** Suppose \((T^\alpha, P)\) is m-f.d. and \(Q : Z \to \Sigma_Q\) is \(\mathcal{H}(T^\alpha, P)\)-measurable. Then \((T^\alpha, Q)\) is m-f.d.

**Proof.** From Lemma 7.2.8, we know that \((T^\alpha, Q \lor P)\) is m-f.d. From Corollary 4.0.33 of the copying lemma, for any \(\delta_0 > 0\) there exists \(\delta\) so that if \((T^\alpha_1, Q_1)\) satisfies

1. \(\|(T^\alpha_1, Q_1), (T^\alpha, Q)\|_* < \delta\) and
2. \(h(T^\alpha_1, Q_1) > h(T^\alpha, Q) - \delta_0/6,\)

then for \((Y, \mathcal{G}, \nu, S)\) a Bernoulli \(G\)-action of entropy at most \(h(T^\alpha, P)\), there is a partition \(P_1\) of \(X_1 \times Y\) satisfying

1. \(\|(T^\alpha_1 \times S, Q_1 \lor P_1), (T^\alpha, Q \lor P)\|_* < \delta_0\) and
2. \(h(T^\alpha_1 \times S, Q_1 \lor P_1) > h(T^\alpha, Q \lor P) - \delta_0.\)

This implies that

\[
\overline{m}(T^\alpha \times S, Q_1 \lor P_1; T^\alpha, Q \lor P) < \varepsilon.
\]

Thus restricting the \(m\)-joinings to the processes \((T^\alpha_1, Q_1)\) and \((T^\alpha, Q)\) we obtain

\[
\overline{m}(T^\alpha_1, Q_1; T^\alpha, Q) < \varepsilon,
\]

completing the proof that \((T^\alpha, Q)\) is m-f.d. \(\square\)

**Definition 7.2.10.** If \((X, \mathcal{F}, \mu, T^\alpha)\) is m-f.d. for all finite partitions \(P\) we say \(T^\alpha\) is m-f.d. Notice we now know this will be implied if \((T^\alpha, P)\) is m-f.d. for a generating partition \(P\).

We now show that m-f.d. is an \(m\)-equivalence invariant.

**Theorem 7.2.11.** Suppose \((X, \mathcal{F}, \mu, T^\beta)\) is m-f.d. Then for any \(\alpha\) with \(\alpha \sim m \beta\), \(T^\alpha\) is also m-f.d.

**Proof.** Fix a partition \(P\) and \(\varepsilon > 0\). Suppose \(\{\phi_i\} \subseteq \Gamma\) with \(\alpha \phi_i \to m \beta\). Set \(\hat{\beta} = \langle \phi_i \rangle_\alpha\) in the \(m_\alpha\)-closure of \(\Gamma\). By Theorem 4.0.27 we can assume all the \((\alpha, \phi_i)\) are bounded rearrangements. Choose \(\phi_I\) so that

\[
m_\alpha(\phi_I, \hat{\beta}) < \varepsilon/3.
\]
7.2. THE $m$-DISTANCE AND $m$-FINITELY DETERMINED PROCESSES

Letting $\hat{\alpha} = \langle \phi^{-1}_i \rangle_{\beta}$ in the $m_{\beta}$-closure of $\Gamma$, we have

$$m_{\beta}(\phi^{-1}_i, \hat{\alpha}) = m_{\alpha}(\phi_i, \hat{\beta}) < \varepsilon / 3.$$ 

As $T^{\beta \phi^{-1}_i}$ is conjugate to $T^\beta$, it is $m$-f.d. In particular $(T^{\beta \phi^{-1}_i}, \mu)$ is an $m$-f.d. process. Hence there exists $\delta > 0$ so that for any process $(X_1, \mathcal{F}_1, \mu_i, T^{\alpha_1}_1, P_1)$ satisfying

1. $\|(T^{\alpha_1}_1, P_1), (T^{\beta \phi^{-1}_i}, P)\|_* < \delta$ and

2. $h_m(T^{\alpha_1}_i P_1) > h_m(T^{\beta \phi^{-1}_i}, P_1) - \delta$ then

3. $\bar{m}(T^{\alpha_1}_1, P_1; T^{\beta \phi^{-1}_i}, P) < \varepsilon_1$.

As $m_{\alpha}(\text{id}, \beta \phi^{-1}_i) < \varepsilon / 3$, there are $\psi_i \in FG(\Omega)$ with $\alpha \psi_i \rightarrow \beta \phi^{-1}_i$ and $\sup_i m(\alpha, \psi_i) < \varepsilon / 3$.

By Lemma 5.0.46 we know

$$\liminf_{i \to \infty} h_m(T^{\alpha \psi_i}, P) \geq h_m(T^{\beta \phi^{-1}_i}, P),$$

(remember, $h_m = h$ as we are assuming the size is entropy-preserving) and certainly

$$\|(T^{\alpha \psi_i}, P), (T^{\beta \phi^{-1}_i}, P)\|_* \to 0.$$

Select $\psi$ from among the $\psi_i$ so that

1. $\|(T^{\alpha \psi}, P), (T^{\beta \phi}, P)\|_* < \delta / 2$ and

2. $h_m(T^{\alpha \psi}, P) \geq h_m(T^{\beta \phi}, P) - \delta / 2$.

Let $Q = P \lor P \circ \psi^{-1}$, and as $(T^{\alpha \psi}, P)$ and $(T^{\alpha}, P \circ \psi^{-1})$ are identical in distribution,

$$h(T^{\alpha}, Q) \geq h(T^{\alpha \psi}, P).$$

Choose a value $\delta_1$ so that if

$$\|(\alpha_1, \psi'), (\alpha, \psi)\|_* < \delta_1$$
then we still will have

$$m(\alpha_1, \psi') < \varepsilon/3.$$  

We proceed now as if $m$ were an entropy preserving size. If $m$ is entropy free, just replace the use of Corollary 4.0.36 with Corollary 4.0.34, without the need of the extra Bernoulli factor to obtain entropy.

By Corollary 4.0.36 there is a $\delta_0$ so that if

1. $\| (T_{\alpha_1}^{\alpha_1}, P), (T^{\alpha}, P) \|_* < \delta_0$ and

2. $h(T_{\alpha_1}^{\alpha_1}, P) \geq h(T, P) - \delta_0$

then for $(Y, \mathcal{G}, \nu, S)$ a Bernoulli shift of sufficient entropy, there is a $\psi'$ in the full group of $U^{\alpha_1} = T_{\alpha_1}^{\alpha_1} \times S$ so that:

3. $\| (U^{\alpha_1 \psi'}, P_1), (T^{\alpha\psi}, P) \|_* < \delta/2$

4. $h(U^{\alpha_1 \psi'}, P_1) \geq h(T^{\alpha\psi}, P) - \delta/2$ and

5. $\| (\alpha_1, \psi'), (\alpha, \psi) \|_* < \delta_1/2.$

From this we can conclude that

$$\overline{m}(U^{\alpha_1 \psi'}, P_1; T_{\alpha_1}^{\beta\psi_1}, P) < \varepsilon/3$$

and so

$$\overline{m}(T_{\alpha_1}^{\alpha_1}; P_1; T_{\beta\psi_1}^{\alpha}, P) < 2\varepsilon/3$$

as $m(\alpha_1, \psi') < \varepsilon/3.$

Hence

$$\overline{m}(T_{\alpha_1}^{\alpha_1}; P_1; T^{\alpha}, P) < \varepsilon$$

as $m(\alpha, \overline{\beta\psi_1}) < \varepsilon/3$, completing the proof that $T^{\alpha}$ is m-f.d. with respect to any partition $P$.  \( \square \)
7.3. The Equivalence Theorem

We now develop the background for and prove the equivalence theorem, that any two \( m \)-finitely determined \( G \)-actions of the same \( m \)-entropy are \( m \)-equivalent. Let \( (X, \mathcal{F}, \mu, T^\alpha) \) and \( (X_1, \mathcal{F}_1, \mu_1, T_1^\alpha) \) be two free and ergodic \( G \)-actions. We already know that the space of measures \( J_m(T_1^\alpha, T^\alpha) \) is a Polish topological space in the weak* topology. What we intend to show is that if both of these actions are \( m \)-finitely determined and they have the same \( m \)-entropy then in fact those \( m \)-joinings that arise from \( m \)-equivalences between the two actions are a residual (dense \( G_\delta \)) subset of \( J_m(T_1^\alpha, T^\alpha) \).

Stated in this form, the equivalence theorem will follow directly from a corresponding “Sinai’s Theorem” whose structure we now develop. Consider the subset \( \mathcal{E}_m \subseteq J_m(T_1^\alpha, T^\alpha) \) defined as follows:

\[
\mathcal{E}_m(T_1^\alpha, T^\alpha) = \{ \mu \in J_m(T_1^\alpha, T^\alpha) : \text{with respect to } \hat{\mu}, \text{ all } f_i \text{ and hence } \phi_i \text{ are } p_1(\mathcal{F}_i) \text{-measurable and} \]

\[
p(\mathcal{F}) \subseteq p_1(\mathcal{F}_i) \}.
\]

For \( \hat{\mu} \) to belong to \( \mathcal{E}_m \) means \( \hat{\mu} \) can be thought of as follows. There are elements in the full-group of \( T_1^\alpha \) for which

\[
(f^{\alpha_1, \phi_1} \otimes f^{\alpha_2, \phi_2} \otimes \ldots) \mu_1 \in \mathcal{M}^m
\]

and hence \( \alpha_1 \phi_i \to \beta \), and there is a measure preserving map \( \eta : X_1 \to X \) with \( \eta T_1^\beta = T^\alpha \eta \). This really is, very loosely speaking, no more than saying \( \alpha_1 \) is \( m \) equivalent to an arrangement \( \beta \) for which \( T_1^\beta \) has \( T^\alpha \) as a factor. To actually obtain the joining in \( \mathcal{E}_m \) one must select a sequence of full-group elements that achieves the \( m \)-equivalence, and then perhaps drop to a subsequence to guarantee they map \( \mu_1 \) into \( \mathcal{M}^m \). To see that any \( \hat{\mu} \in \mathcal{E}_m \) gives rise to this situation, just notice that from the definition of \( \mathcal{E}_m \), one can project the full-group elements \( \phi_i \) from \( X_1^G \times X^G \times \bar{\mathcal{R}} \) to \( X_1 \), as well as the identification of \( \mathcal{F} \) as a subalgebra of \( \mathcal{F}_1 \).

Notice further that to have both \( \hat{\mu} \in \mathcal{E}_m(T_1^\alpha, T^\alpha) \) and \( \hat{\mu}^\prime \in \mathcal{E}_m(T^\alpha, T_1^\alpha) \) is to say that \( \hat{\mu} \) arises from a pair of \( m \)-equivalent arrangements whose corresponding free and ergodic \( G \)-actions are conjugate to \( T_1^\alpha \) and \( T^\alpha \) respectively, i.e. the two actions are \( m \)-equivalent, and \( \hat{\mu} \) is a detailed description of one such \( m \)-equivalence between them.

What we want to show first is that \( \mathcal{M}_m(T_1^\alpha, T^\alpha) \) is a \( G_\delta \) subset of \( J_m(T_1^\alpha, T^\alpha) \) (perhaps empty). To complete the equivalence theorem what we will show is that if \( T^\alpha \) is \( m \)-f.d., and \( h_m(T_1^\alpha) \geq h_m(T^\alpha) \), then \( \mathcal{E}_m(T_1^\alpha, T^\alpha) \) is a dense subset of \( J_m(T_1^\alpha, T^\alpha) \).
7. THE EQUIVALENCE THEOREM

To that end, for $I \in \mathbb{N}$, $P : X \to \Sigma_P$, a finite partition, and $\varepsilon > 0$ let
\[
\mathcal{O}(I, P, \varepsilon) = \{ \hat{\mu} \in J_m(T_1^\alpha, T^\alpha) : \\
i) \text{ there are } \phi'_i \quad i = 1, \ldots, I \text{ in the full-group of } T_1^\alpha \text{ with } \\
\hat{\mu}(\{y = (p_i(x_1), p(x), \{f_i\}) : \phi'_i(x_1) \neq \phi_i(y)} \} < \varepsilon \text{ and } \\
\text{ii) there is a } P' : X_1 \to \Sigma_P \text{ with } \hat{\mu}(\{(p_i(x_1), p(x), \{f_i\}) : \\
P'(x_1) \neq P(x)} \}) < \varepsilon \}
\]

That is to say, an $m$-joining belongs to $\mathcal{O}(I, P, \varepsilon)$ if, loosely speaking, the first $I$ full group elements $\phi_i$ in the joining can be approximated by full-group elements measurable with respect to $\mathcal{F}_i$ (the first coordinate algebra) and the partition $P$, which is measurable with respect to $\mathcal{F}$ (the second coordinate $\sigma$-algebra) can be approximated by a partition in $\mathcal{F}_i$ (the first coordinate algebra).

**Lemma 7.3.1.** The sets $\mathcal{O}(I, P, \varepsilon)$ are open subsets of $J_m(T_1^\alpha, T^\alpha)$.

**Proof.** Let $\hat{\mu} \in \mathcal{O}(I, P, \varepsilon)$. As the topological space $J_m(T_1^\alpha, T^\alpha)$ is a conjugacy invariant of the two $G$-actions, we can assume that the elements $\phi'_i$, $i = 1, \ldots, I$ and the partition $P'$ are all continuous functions of $X_1$ (in detail this is simply asking that a countable collection of sets be assumed clopen). Similarly we can assume that $P$ is a continuous function of $X$. Now both $i)$ and $ii)$ of the definition of $\mathcal{O}(I, P, 1/j)$ are simply asking that the $L^1(\hat{\mu})$ norms of a finite collection of continuous characteristic functions are $< \varepsilon$. Thus there will be a neighborhood of $\hat{\mu}$ in the weak* topology on which both $i)$ and $ii)$ will continue to hold. \hfill $\square$

**Lemma 7.3.2.** Let $P_i$ be a countable collection of finite Borel partitions of $X$, dense in the space of all Borel partitions taking values in $\mathbb{N}$. Then
\[
\mathcal{E}_m(T_1^\alpha, T^\alpha) = \bigcap_{i,j \in \mathbb{N}} \mathcal{O}(I, P_i, 1/j),
\]
and hence is a $G_{\delta}$.

**Proof.** One containment follows directly from the definition, that
\[
\mathcal{E}_m(T_1^\alpha, T^\alpha) \subseteq \bigcap_{i,j \in \mathbb{N}} \mathcal{O}(I, P_i, \mu).
\]
To see the other containment, let $\hat{\mu} \in \bigcup_j \mathcal{O}(I, P_i, 1/j)$. There will be a partition $P'_i$ of $X$ and elements $\phi'_i$, $i = 1, \ldots, I$ in the full-group of $T_1^\alpha$ with
\[
P'_i = P_i, \quad \hat{\mu}\text{-a.s. and } \\
\phi'_i = \phi_i, \quad i = 1, \ldots, I, \quad \hat{\mu}\text{-a.s.}
\]
Hence for all $\mu \in \cap_{i,j} \mathcal{O}(I, P_i, 1/j)$,

$$P'_i = P_i, \quad \mu\text{-a.s. and}$$
$$\phi'_i = \phi_i, \quad \mu\text{-a.s. for all } i.$$

As the collection of sets in $p(\mathcal{F})$ that are also $\mu$-a.s. also in $p_1(\mathcal{F}_1)$ forms a $\mu$-complete $\sigma$-algebra which we have just shown contains all of the $P_i$, $p(\mathcal{F})$ is $\mu$-a.s. a sub-$\sigma$-algebra of $p_1(\mathcal{F}_1)$. This proves the other containment

$$\cap_{i,j} \mathcal{O}(I, P_i, 1/j) \subseteq \mathcal{E}_m(T^{\alpha_1}, T^{\alpha}).$$

$\square$

To complete the equivalence theorem we show that if $(X, \mathcal{F}, \mu, T^{\alpha})$ is an $m$-f.d. free and ergodic $G$-action, then for any $(X_1, \mathcal{F}_1, \mu_1, T_1^{\alpha_1})$ with $h_m(T_1^{\alpha_1}) \geq h_m(T^{\alpha})$, then each of the sets $\mathcal{O}(I, P, \varepsilon)$ is in fact dense in $J_m(T_1^{\alpha_1}, T^{\alpha})$. Having already shown in Theorem 6.5.8 that $J_m(T_1^{\alpha_1}, T^{\alpha})$ is a Polish space, we immediately conclude that $\mathcal{E}_m(T_1^{\alpha_1}, T^{\alpha})$ is a dense subset of this space $m$-joinings. We refer to this proof of denseness as the “Sinai’s Theorem” of our work as it embeds the $m$-f.d. action $T_1^{\alpha_1}$ as a factor of some $T^\beta$ where $\alpha \sim \beta$.

**Theorem 7.3.3.** Suppose $(X, \mathcal{F}, \mu, T^{\alpha})$ is an $m$-f.d. free and ergodic $G$-action, and $(X_1, \mathcal{F}_1, \mu_1, T_1^{\alpha_1})$ is a free and ergodic $G$-action with

$$h_m(T_1^{\alpha_1}) \geq h_m(T^{\alpha}).$$

Then for any $I \in \mathbb{N}$, finite partition $P$ and $\varepsilon > 0$, the set $\mathcal{O}(I, P, \varepsilon)$ is dense in $J_m(T_1^{\alpha_1}, T^{\alpha})$.

**Proof.** Let $\tilde{\mu} \in J_m(T_1^{\alpha_1}, T^{\alpha})$ be some fixed $m$-joining, and let $\eta$ be an open neighborhood of $\tilde{\mu}$ in $J_m(T_1^{\alpha_1}, T^{\alpha})$. Notice that in $J_m(T_1^{\alpha_1}, T_2^{\alpha_2})$ the arrangement $\alpha$ represents $\alpha_1$ and $\beta$ represents $\alpha$. We need to show that $\mathcal{O}(I, P, \varepsilon) \cap \eta$ is not empty.

As $J_m(T_1^{\alpha_1}, T^{\alpha})$ is a conjugacy invariant, we can assume both $X$ and $X_1$ are 0-dimensional spaces, that is to say have dense families of clopen sets, including the $P$-measurable sets. Hence we can find partitions $Q$ of $X_1$ and $P'$ of $X$, $P'$ a refinement of $P$, a value $I_0 \in \mathbb{N}$ and a value $\varepsilon_0$, $\varepsilon > \varepsilon_0 > 0$ so that if $\tilde{\mu}_1 \in J_m(T_1^{\alpha_1}, T^{\alpha})$ satisfies
(a) \[ \| (\sigma^\alpha, Q \circ \pi_1 \lor P' \circ \pi_2 \lor \bigvee_{i=1}^{I_0} g(\alpha, \phi_1)) \|_* < \epsilon_0 \]

then \( \hat{\mu}_1 \in \eta \). We will construct \( \hat{\mu}_1 \) satisfying this as well as

(b) \( \hat{\mu}_1 \in \O(I_0, P', \epsilon_0) \subseteq \O(I, P, \epsilon) \).

To obtain \( \hat{\mu}_1 \) we will construct an \( I_0, J, \delta \)-perturbation of the free and ergodic \( G \)-action

\((X_1 \times X)^G \times \tilde{\mathcal{K}}, \mathcal{B}, \hat{\mu}, \sigma^\alpha)\).

This perturbation will be the \( m \)-joining we want in the loose sense. We then just map it by \( Q^* \) into the space of \( m \)-joinings.

By Theorem 7.1.5 there exists \( \delta_1 > 0 \) such that for any \( J_1 \) sufficiently large, any \( I_0, J_1, \delta_1 \)-perturbation of the above system will still project to a measure in \( \mathcal{M}^m \). Our choices for \( \delta \) and \( J \) will be at least this small and large respectively, and hence all we need see is how to also obtain both (a) and (b) above via a perturbation.

As \( (T^\alpha, P') \) is \( m \)-f.d., there is a \( \delta_2 > 0 \) so that if \( T_2^{\alpha_2} \) is some free and ergodic \( G \)-action with:

1. \( \|(T_2^{\alpha_2}, P'_2), (T^\alpha, P')\|_* < \delta_2 \) and
2. \( h_m(T_2^{\alpha_2}) > h_m(T^\alpha, P') - \delta_2 \) then
3. \( m(T_2^{\alpha_2}, P'_2; T^\alpha, P') < \delta_1 \).

As \( \alpha \phi_1 \to \beta \) in the \( m \)-joining \( \hat{\mu} \) and \( (\sigma^\beta, P') \) is identical to \( (T^\alpha, P') \) in distribution, we must have both

1. \( \|(\sigma^{\alpha \phi_1}, P'), (T^\alpha, P')\|_* \to 0 \) and
2. \( h_{m, \hat{\mu}}(\sigma^{\alpha \phi_1}, P') \to h_m(T^\alpha, P') \).

Select \( J_1 \) larger if necessary to ensure:

1. \( \|(\sigma^{\alpha \phi_{J_1}}, P'), (T^\alpha, P')\|_* < \delta_2 / 2 \) and
2''. \( h_{m,\hat{\mu}}(\sigma^{\alpha_1 \phi_{h_1}}, P') > h_m(T^\alpha, P') - \delta_2/2. \)

As \( h_m(T_1^{\alpha_1}) \geq h_m(T^\alpha) \), select a finite partition \( Q'' \) of \( X_1 \) refining \( Q' \) with

\[
h_m(T_1^{\alpha_1}, Q'') > h_m(T^\alpha) - \delta_2/3.
\]

Consider the free and ergodic \( G \)-action \( \sigma^\alpha \) relative to the measure \( \hat{\mu} \) and partitions \( Q'', P' \) and the finite list of full group elements \( \phi_1, \phi_2, \ldots, \phi_{t_h}, \phi_{J_1} \). We are precisely in the position to apply Theorem 4.0.37 to copy in all of these full-group elements along with the partition \( P' \). Construct these copies sufficiently close in distribution and in \( m \)-entropy (which here is of course just entropy) to obtain all of the following.

i) \( D((f^{\alpha, \phi_1} \otimes \cdots \otimes f^{\alpha, \phi_{t_h}}) \otimes f^{\alpha, \phi_{J_1}})^* \hat{\mu},
\[
\|D(f^{\alpha, \phi_1} \otimes \cdots \otimes f^{\alpha, \phi_{t_h}}) \otimes f^{\alpha, \phi_{J_1+1}})^* \mu_1) < \delta_1
\]

1\(^{(3)} \). \( \|((T^{\alpha_1 \phi_{t_h+1}} P_1), (T^\alpha, P'))_s < \delta_2 \) from \( 1'' \).

2\(^{(3)} \). \( h_m(T_1^{\alpha_1 \phi_{t_h+1}}, P_1) > h_m(T^\alpha, P') - \delta_2 \) and hence we will have

3\(^{(3)} \). \( m(T_1^{\alpha_1 \phi_{t_h+1}}, P_1; T^\alpha, P') < \delta_1. \)

We also require of the copying that

\[
(a)' \quad \|((\sigma^\alpha, Q'' \circ \pi_1 \vee P' \circ \pi_2 \vee \bigvee_{i=1}^{t_h+1} g(\alpha, \phi_i))_{\hat{\mu}},
\[
\|((T_1^{\alpha_1}, Q'' \circ P' \vee \bigvee_{i=1}^{t_h+1} g(\alpha, \phi_i))_s < \varepsilon_0.
\]

Now 3\(^{(3)} \) tells us there is an \( m \)-joining \( \hat{\mu}_0 \) of \( T_1^{\alpha_1 \phi_{t_h+1}} \) and \( T^\alpha \) with

1. \( \hat{\mu}_0(P_1 \triangle P') < \delta_1 < \varepsilon_0 \) and

2. \( m(\hat{\mu}_0) < \delta_1. \)

We can lift to this \( m \)-joining of \( T_1^{\alpha_1 \phi_{t_h+1}} \) and \( T^\alpha \) the full-group elements \( \phi'_i \), and the arrangement \( \alpha_1 \) yielding the sequence of \( G \)-actions
\[ \sigma^{\alpha_1}, \sigma^{\alpha_1 \phi'_1}, \ldots, \sigma^{\alpha_1 \phi'_{h+1}} \text{ where } \alpha = \alpha_1 \phi'_{h+1}. \]

The \( m \)-joining now comes equipped with its canonical full-group elements \( \phi_i \) with \( \alpha \phi_i \rightarrow \beta \) and for which

\[
\limsup_{i \to \infty} m_{\hat{\mu}_0}(\alpha, \phi_i) < \delta_1
\]

which is to say

ii) \( m_{\alpha_1 \phi_{h+1} + \hat{\mu}_0}(\langle \phi_i \rangle, \text{id}) < \delta_1 \).

Notice that i) and ii) tell us that the \( G \)-action \( ((X \times X)^G \times \bar{R}, B, \hat{\mu}_0, \sigma^{\alpha_1}) \) with full group elements \( \phi'_1, \ldots, \phi'_{h+1} \) and \( \{\phi_i\} \) form an \( I_0, J_1, \delta_1 \)-perturbation of the original \( m \)-joining \( \hat{\mu}_1 \) with its canonical list of full-group elements. In particular we know by Lemma 7.1.5 that dropping to a subsequence of the \( \phi_i \) if necessary we will have

\[
(\sigma^{\alpha_1} \phi'_1 \otimes \sigma^{\alpha_1} \phi'_2 \otimes \ldots) ^* \hat{\mu}_0 \in \mathcal{M}^m.
\]

Examining this perturbation more closely, consider the two projections:

\[
\pi_1(\{\bar{x}\}, \{\bar{x}_1\}, \{f_i\}) \rightarrow \bar{x}(\text{id}) \quad \text{and}
\]

\[
\pi_2(\{\bar{x}\}, \{\bar{x}_1\}, \{f_i\}) \rightarrow \bar{x}_1(\text{id}).
\]

These satisfy:

\[
\pi_1^*(\hat{\mu}_0) = \mu_1, \quad \pi_2^*(\hat{\mu}_0) = \mu,
\]

\[
\pi_1 \sigma^{\alpha_1} = T^{\alpha_1} \pi_1, \quad \text{and} \quad \pi_2 \sigma^{\beta} = T^{\alpha} \pi_2
\]

and hence this is an \( m \)-joining of \( T^{\alpha_1} \) and \( T^{\alpha} \) in the loose sense. Let

\[
\hat{\mu}_1 = Q^*(\hat{\mu}_0)
\]

be the corresponding \( m \)-joining in \( J_m(T^{\alpha_1}, T^{\alpha}) \).

Notice that \( \phi_i^T = \phi'_i \) for \( i = 1, \ldots, I_0 \) are all \( \pi_1^{-1} \mathcal{F}_1 \)-measurable, and as \( \hat{\mu}_0(P_1 \Delta P') < \varepsilon_0 \), we will have (b), that

\[
\hat{\mu}_1 \in \mathcal{O}(I_0, P', \varepsilon_0).
\]

Statement (a') implies directly that \( \hat{\mu}_1 \) satisfies (a) and the result is proven. \( \square \)