Definitions and Examples

2.1. Orbits, Arrangements and Rearrangements

Let \((X, \mathcal{F}, \mu)\) be a fixed nonatomic Lebesgue probability space. Let \(G\) be an infinite discrete amenable group. Let \(\mathcal{O} \subseteq X \times X\) be an ergodic, measure preserving, hyperfinite equivalence relation. For our purposes, this simply means that \(\mathcal{O} = \{(x, T_g(x))\}_{g \in G}\) where \(T : G \times X \rightarrow G\) (written of course \(T_g(x)\)) is some ergodic and free, measure preserving action of \(G\) on \(X\).

**Definition 2.1.1.** Let \(G\) be an infinite countable discrete amenable group. A \textbf{G-arrangement} \(\alpha\) is any map from \(\mathcal{O}\) to \(G\) that satisfies:

(i) \(\alpha\) is 1-1 and onto, in that for a.e. \(x \in X\), for all \(g \in G\), there is a unique \(x' \in X\) with \(\alpha(x, x') = g\). We write \(x' = T^\alpha_g(x)\);

(ii) \(\alpha\) is measurable and measure preserving, i.e. for all \(A \in \mathcal{F}, g \in G\), both \(T^\alpha_g(A) \in \mathcal{F}\) and \(\mu(T^\alpha_g(A)) = \mu(A)\); and

(iii) \(\alpha\) satisfies the cocycle equation \(\alpha(x, x_3)\alpha(x_1, x_2) = \alpha(x_1, x_3)\).

As \(G\) will not vary for our considerations we will abbreviate this as an arrangement. Let \(\mathcal{A}\) denote the set of all such arrangements.

**Lemma 2.1.2.** \(\alpha\) is a \(G\)-arrangement if and only if there is a measure preserving ergodic free action of \(G\), \(T\), whose orbit relation is \(\mathcal{O}\) such that \(\alpha(x, T_g(x)) = g\) for all \((x, T_g(x)) \in \mathcal{O}\).

Thus the vocabulary of \(G\)-arrangements on \(\mathcal{O}\) is precisely equivalent to the vocabulary of \(G\)-actions whose orbits are \(\mathcal{O}\). For a \(G\)-arrangement \(\alpha\), we write \(T^\alpha\) for the corresponding action. For a \(G\)-action \(T\), we write \(\alpha_T\) for the corresponding \(G\)-arrangement.

**Definition 2.1.3.** The full group of \(\mathcal{O}\) is the group (under composition) \(\Gamma\) of all measure preserving invertible maps \(\phi : X \rightarrow X\) such that for \(\mu\)-a.e. \(x \in X\), \((x, \phi(x)) \in \mathcal{O}\).
Note that it would be sufficient in this definition to assume that $\phi$ is measurable and 1-1, since the fact that $\mathcal{O}$ is a measure preserving orbit relation forces $\phi$ to be measure preserving. Also note that the orbits of $\phi$ are a subrelation of $\mathcal{O}$ and need not be all of $\mathcal{O}$.

**Definition 2.1.4.** A G-rearrangement of $\mathcal{O}$ is a pair $(\alpha, \phi)$, where $\alpha$ is a G-arrangement of $\mathcal{O}$ and $\phi \in \Gamma$. As $G$ is fixed for our purposes we will abbreviate this as a rearrangement. Let $\mathcal{Q}$ denote the set of all such rearrangements.

Intuitively, a rearrangement is simply a change (i.e. rearrangement) of an orbit from the arrangement $\alpha$ to the arrangement $\alpha \phi$, where $\alpha \phi(x, x') = \alpha(\phi(x), \phi(x'))$. One can formalize such a rearrangement in three different ways. Set $B$ to be the set of bijections of $G$ and $B$ the subgroup of $G$ fixing the identity. Both are topologized via the product topology on $G^G$. Notice there is a homomorphism $H : B \rightarrow G$ given by $H(q)(g) = q(id^{-1}q(g))$. Observe information is lost in mapping via $H$. The kernel of $H$ consist of the left translation maps.

To a rearrangement we can associate a family of functions $q^\alpha \phi \in B$ where

$$q^\alpha \phi_x(g) = \alpha(x, \phi(T^\alpha_g(x))).$$

Now suppose $\alpha$ and $\beta$ are two arrangements of the orbits $\mathcal{O}$. Regard the first as an initial arrangement and the second as a terminal arrangement. We can associate to this pair and any point $x$ a bijection from $G$ fixing the identity that describes how the arrangement of the orbit has changed:

$$h^\alpha \beta_x(g) = \alpha(x, T^\alpha_g(x)).$$

Notice here that $\tilde{H}(q^\alpha \phi) = h^\alpha \alpha \phi.$

Write $h^\alpha \beta : X \rightarrow G$.

The third way to view a rearrangement pair has a symbolic dynamic flavor. For each orbit $\mathcal{O}(x) = \{x' \in \mathcal{O} \mid (x, x') \in \mathcal{O} \}$, a rearrangement $(\alpha, \phi)$ also gives rise to a natural map $G \rightarrow G$ (not a bijection though), given by

$$f^\alpha \phi_x(g) = \alpha(T^\alpha_g(x), \phi(T^\alpha_g(x))$$

Visually, regarding $\mathcal{O}(x)$ laid out by $\alpha$ as a copy of $G$, $\phi$ translates the point at position $g$ to position $f^\alpha \phi_x(g)g$. Notice in particular that the definition of $f^\alpha \phi$ is stationary in that

$$f^\alpha \phi_x(g) = f^\alpha \phi_{T^g_x}(id).$$
Thus if we map a point \( x \) to the infinite word
\[
w(x) = \{f_x^{\alpha, \phi}(g)\}_{g \in G} \in G^G
\]
then
\[
w(T_g(x)) = \sigma_g(w(x))
\]
where \( \sigma_g \) is the **shift** action on \( G^G \),
\[
\sigma_g(h)(k) = h(kg).
\]

There is a natural link between the three functions \( h^{\alpha, \phi}, q^{\alpha, \phi} \) and \( f^{\alpha, \phi} \) as follows. For any map \( f : G \to G \) we define
\[
Q(f)(g) = f(g)g \quad \text{and} \quad H(f)(g) = f(g)gf(id)^{-1}.
\]

It is an easy calculation that
\[
H(f^{\alpha, \phi}) = h^{\alpha, \phi} \quad \text{and} \quad Q(f^{\alpha, \phi}) = q^{\alpha, \phi}.
\]

Let \( \{F_i\} \) be a fixed Følner sequence for \( G \). We will describe a number of concepts in terms of the \( F_i \), but will note at appropriate points where they are, in fact, independent of the Følner sequence.

**Lemma 2.1.5.** Let \( \{F_i\} \) be a Følner sequence in \( G \), and let \((\alpha, \phi)\), be a rearrangement. Define
\[
b_i(x) = \#\{g \in F_i; f_x^{\alpha, \phi}(g)g \notin F_i\} = \#\{g \notin F_i; f_x^{\alpha, \phi}(g)g \in F_i\}.
\]
Then
\[
\lim_{i \to \infty} \frac{\|b_i(x)\|}{\#F_i} = 0.
\]

**Proof.** For \( \varepsilon > 0 \), choose a finite set \( K \subseteq G \) so that \( E_K = \{x; f_x^{\alpha, \phi}(id) \notin K\} \) satisfies \( \mu(E_K) < \varepsilon/4 \). Let \( h_i(x) = \#\{g \notin F_i; T_g^{\alpha}(x) \in E_K\} \). By the \( L^2 \)-ergodic theorem, there exists \( I \), such that for \( i \geq I \),
\[
\frac{\|h_i(x)\|}{\#F_i} < \varepsilon/2.
\]

By the Følner property, we may further select \( I \) such that for \( i \geq I \),
\[
\frac{\#\bigcup_{k \in K}(kF_i \triangle F_i)}{\#F_i} < \varepsilon/2.
\]

Now for \( i \geq I \), select \( g \in F_i \) such that \( f_x^{\alpha, \phi}(g)g \notin F_i \). For such \( g \), either
1. \( f_x^{\alpha, \phi}(g) = f_x^{\alpha, \phi}(id) \notin K \); or
2. \( f_x^{\alpha, \phi}(g)g = f_x^{\alpha, \phi}(id)g \in \bigcup_{k \in K}(kF_i \triangle F_i) \).
Thus \( b_i(x) \leq h_i(x) + \# \bigcup_{k \in K}(kF_i \Delta F_i) \), so that \( \limsup_{i \to \infty} \left\| \frac{b_i(x)}{h_i(x)} \right\|_2 < \varepsilon \), which completes the proof. \( \square \)

We now consider three pseudometrics on the set of rearrangements. These all arise from natural topologies on functions \( G \to G \), that is to say on \( G^G \). As \( G \) is countable the only reasonable topology is the discrete one, using the discrete 0,1 valued metric. This topologizes \( G^G \) as a metrizable space with the product topology. This is the weakest topology for which the evaluations \( g : f \to f(g) \) are continuous functions. Notice that \( H \) is a continuous map from \( G^G \) to itself and the map \( h \to h^{-1} \) on \( G \) is continuous.

Define a metric \( d \) on \( G \) as follows. List the elements of \( G \) as \( \{g_1 = \text{id}, g_2, \ldots \} \) and let \( d_0 \) be the 0,1 valued metric on \( G \). Set

\[
d(h_1, h_2) = \sum_i [d_0(h_1(g_i), h_2(g_i)) + d_0(h_1^{-1}(g_i), h_2^{-1}(g_i)))]2^{-(i+1)}.
\]

Notice that if \( h_1, h_2, h_1^{-1}, \) and \( h_2^{-1} \) agree on \( g_1, \ldots, g_i \) then \( d(h_1, h_2) \leq 2^{-i} \). On the other hand if \( d(h_1, h_2) < 2^{-i} \) then \( h_1, h_2 \) and their inverses agree on this list of \( i \) terms.

**Lemma 2.1.6.** The metric \( d \) on \( G \) gives the restricted product topology and makes \( G \) a complete metric space.

**Proof.** The only piece not evident from the above remarks is that \( d \) makes \( G \) complete. Suppose \( h_i \) are a \( d \)-Cauchy sequence. It follows that for all \( g \in G \), for all \( i \) sufficiently large, both \( h_i(g) \) and \( h_i^{-1}(g) \) remain constant. Hence \( h_i \to h \in G^G \) and \( h_i^{-1} \to k \in G^G \). But for any \( i \) sufficiently large

\[
h_i(k(g)) = g
\]

and then clearly \( h \circ k = \text{id} \) and \( h \in G \). \( \square \)

A simple corollary of this is that \( G \) is a \( G_\delta \) subset of \( G^G \) as we have just seen it to be topologically complete in the product topology.

We can use this to define an \( L^1 \) metric on arrangements:

\[
\| \alpha, \beta \|_1 = \int d(h_\alpha \circ \beta, \text{id}) \, d\mu.
\]

As \( d(h_1, h_2) = d(h_2^{-1} h_1, \text{id}) \) and \( (h_\alpha \circ \beta)^{-1} = h_\beta \circ \alpha \) we see that this is a metric.

**Lemma 2.1.7.** The metric space \( (A, \| \cdot ; \cdot \|_1) \) is complete.
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**Proof.** Suppose \( \alpha_i \) form a Cauchy sequence of arrangements in \( \| \cdot \|_1 \). There will exist then a subsequence \( i(j) \) converging pointwise, i.e. so that for \( \mu \)-a.e. \( x \in X \), \( h_x^{\alpha_{i(j)}} \) are Cauchy in the metric \( d \). As \( (G, d) \) is complete, we conclude that for \( \mu \)-a.e. \( x \in X \),

\[
h_x^{\alpha_{i(j)}} \xrightarrow{j} h_x.
\]

Setting

\[
\beta(x, T_g^\alpha(x)) = h_x(g)
\]

it is straightforward that \( \beta \) is an arrangement. As \( \| \alpha_{i(j)} \|_1 \to 0 \) we are done. \( \Box \)

We can also define a metric similar to \( d \) on \( G^G \) itself making it a complete metric space by just taking half of the terms in \( d \):

\[
d_1(f_1, f_2) = \sum_i d_0(f_1(g_i), f_2(g_i)) 2^{-i}.
\]

This also leads to an \( L^1 \) metric on \( G^G \)-valued functions on a measure space:

\[
\| f_1, f_2 \|_1 = \int d_1(f_1, f_2) d\mu.
\]

These two \( L^1 \) distances now give us two families of \( L^1 \) distances on the full-group, one a metric the other a pseudometric, associated with an arrangement \( \alpha \):

\[
\| \phi_1, \phi_2 \|_w^\alpha = \int d(h^{\alpha, \phi_1}, h^{\alpha, \phi_2}) d\mu
\]

\[
= \int d(h^{\alpha \phi_1, \alpha \phi_2}, \text{id}) d\mu = \| \alpha \phi_1, \alpha \phi_2 \|_1
\]

\[
= \int d(H(f^{\alpha, \phi_1}), H(f^{\alpha, \phi_2}) d\mu,
\]

and

\[
\| \phi_1, \phi_2 \|_s^\alpha = \int d_1(f^{\alpha, \phi_1}, f^{\alpha, \phi_2}) d\mu
\]

\[
= \| f^{\alpha, \phi_1}, f^{\alpha, \phi_2} \|_1
\]

The weak \( L^1 \) distance, \( \| \cdot, \|_w^\alpha \), is only a pseudometric but the strong \( L^1 \) distance, \( \| \cdot, \|_s^\alpha \), is a metric. Notice that \( T \)-invariance of \( \mu \) tells us
\[
\|\phi_1, \phi_2\|_{s}^\alpha \leq 2 \int d_0(f_\alpha^{\phi_1}(id), f_\alpha^{\phi_2}(id)) \, d\mu
= 2\mu(\{x : \phi_1(x) \neq \phi_2(x)\}) \leq 2\|\phi_1, \phi_2\|_{s}^\alpha.
\]

Thus, in fact, the topology generated by the strong \(L^1\) distance on the full-group is independent of the arrangement \(\alpha\).

Before moving on to the weak* pseudometric we point out that it would perhaps be more correct to call the above weak and strong “distribution” metrics as they measure how closely the two rearrangements distribute the mass of \(X\) on the space of functions \(G^G\). We of course call them \(L^1\) pseudometrics as they arise from integrals of “distances”.

To describe the weak*-distance between two arrangements we need let \(G^* = G \cup \{\ast\}\) be the one point compactification of \(G\). Now \((G^*)^G\) is a compact metric space and hence the Borel probability measures on \((G^*)^G\), which we write as \(\mathcal{M}_1(G^*)\), are a compact and convex space in the weak* topology (that is to say the topology induced on Borel measures as the dual of the continuous functions).

We can put an explicit metric on this space as follows. For any finite subset \(F \subset G\) and \(f \in G^G\) let \(f_F\) be the restriction of \(f\) to \(F\). As \(f_F\) can be one of at most a countable collection of values, \(f \mapsto f_F\) partitions \(G^G\) into a countable collection of clopen sets. If two measures on \((G^*)^G\) agree on these sets, that is to say on all cylinder sets that do not have a \(\ast\) in their name, then they agree. Moreover the characteristic functions of these sets are continuous. Hence if \(\mu_i(f_F) \rightarrow \mu(f_F)\) the \(\mu_i \rightarrow \mu\) weak*. To turn this into a metric, let \(F_i\) be an increasing sequence of finite sets that exhaust \(G\), for example a Følner sequence. For each \(F_i\) let \(P(F_i)\) be the partition of \(G^G\) according to the values \(f_{F_i}\). These partitions refine and for any fixed \(F\), once \(F \subset F_i\), \(f_F\) will be \(P(F_i)\)-measurable. Set

\[
D(\mu_1, \mu_2) = \sum_i \left( \sum_{p \in P(F_i)} |\mu_1(p) - \mu_2(p)| \right) 2^{-i+1}.
\]

Notice that

\[
\sum_{p \in P(F_i)} |\mu_1(p) - \mu_2(p)| / 2 \leq 1
\]

and so the \(i\)th term in this sum is bounded by \(2^{-i}\). Moreover as the partitions \(P(F_i)\) refine, the values

\[
\sum_{p \in P(F_i)} |\mu_1(p) - \mu_2(p)| / 2 \leq 1
\]
increase. It follows that for all $i$

$$
\sum_{p \in P(F_i)} |\mu_1(p) - \mu_2(p)| + 2^{-i} \geq D(\mu_1, \mu_2).
$$

Before continuing we make a two remarks. First it is clear from the discussion that this metric gives the weak* topology on $(G^*)^G$. It is not too difficult to argue that those measures which put no support on $\star$ are a $G_\delta$ subset of the measures in $(G^*)^G$. We will take a broader approach to this particular issue later in Section 7, showing that not only are the measures supported on $G^G$ a $G_\delta$ but the weak* topology here is independent of the way we choose to compactify $G^G$ as long as the compactification is metric. At this point these issues are not important.

We now define the distribution pseudometric between two rearrangements by

$$
\|(\alpha, \phi), (\beta, \psi)\|_* = D((f^{\alpha, \phi})^*(\mu), (f^{\beta, \psi})^*(\nu)).
$$

We can combine the two $L^1$-metrics on arrangements and the full-group to define a product metric on rearrangements in the form

$$
\|(\alpha_1, \phi_1), (\alpha_2, \phi_2)\|_1 = \|\alpha_1, \alpha_2\|_1 + \mu(\{x : \phi_1(x) \neq \phi_2(x)\}).
$$

We end this Section by relating this complete $L^1$-metric on rearrangements to the distribution pseudometric.

**Lemma 2.1.8.** The map $(\alpha, \phi) \to (f^{\alpha, \phi})^*(\mu)$ is uniformly continuous as a map from $(\mathcal{Q}, \|\cdot\|_1)$ to $(\mathcal{Q}, \|\cdot\|_*)$. That is to say, given any $\varepsilon > 0$ there exists $\delta > 0$ such that if $(\alpha_1, \phi_1)$ and $(\alpha_2, \phi_2)$ are two $G$-rearrangements that satisfy $\|(\alpha_1, \phi_1), (\alpha_2, \phi_2)\|_1 < \delta$ then

$$
\|(\alpha_1, \phi_2), (\alpha_2, \phi_2)\|_* < \varepsilon.
$$

**Proof.** Let $F \subseteq \{g_1, g_2, \ldots, g_K\}$ be any finite set. Suppose $\delta > 0$ and $\|\alpha_1, \alpha_2\|_1 < d/(K2^K)$. Then

$$
\mu(\{x : h_{x, \alpha_1, \alpha_2}^i(g_i) \neq g_i \text{ for some } i \leq K\} \leq \delta.
$$

Thus,

$$
\mu(\{x; f_{x, F}^{\alpha_1, \phi_1} \neq f_{x, F}^{\alpha_2, \phi_2}\})
\leq \#F \mu(\{x : \phi(x) \neq \psi(x)\})
\quad + \mu(\{x : h_{x, \alpha_1, \alpha_2}^i(g_i) \neq g_i \text{ for some } i \leq K\})
\leq (\#F + K2^K)\|(\alpha_1, \phi_1), (\alpha_2, \phi_2)\|_1
\leq \delta(\#F + K2^K).$$
Thus for all $i$,
\[ \| (\alpha_1, \phi_1) \|_\ast \leq (\# F_i + K_i 2^{K_i}) \delta, \]
where $F_i \subseteq \{ g_1, \ldots, g_{K_i} \}$.

Let $\varepsilon > 0$. Select $i$ so that $2^{-i} < \varepsilon / 2$. Select
\[ \delta = \varepsilon / (2(\# F_i + 2^{K_i} K_i)). \]

The result follows. \hfill \qed

### 2.2. Definition of a size and $m$-equivalence

In this section we define the notion of a **size** $m$ on rearrangements $(\alpha, \phi)$ as a family of pseudometrics $m_\alpha$ on the full-group satisfying some simple relations to the metrics and pseudometrics we defined in the previous section. We then define the $m$ equivalence class of an arrangement $\alpha$ to be those arrangements $\beta$ for which the corresponding $m_\alpha$ and $m_\beta$-completions of the full-group are isometric in a canonical fashion.

As earlier we let $\Gamma$ denote the full group of $\mathcal{O}$. Recall that $\mathcal{A}$ denotes the set of arrangements and $\mathcal{Q}$ denotes the set of rearrangements.

A size is a function
\[ m : \mathcal{Q} \to \mathbb{R}^+ \]
such that, if we write
\[ m_\alpha(\phi_1, \phi_2) = m(\alpha \phi_1, \phi_1^{-1} \phi_2), \]
then $m$ satisfies the following three axioms.

**Axiom 1.** For each $\alpha \in \mathcal{A}$, $m_\alpha$ is a pseudometric on $\Gamma$.

**Axiom 2.** For each $\alpha \in \mathcal{A}$, the identity map
\[ (\Gamma, m_\alpha) \xrightarrow{\text{id}} (\Gamma, \| \cdot \|_\ast) \]
is uniformly continuous.

In particular this means that if $m_\alpha(\phi_1, \phi_2) = 0$ then the two arrangements $\alpha \phi_1$ and $\alpha \phi_2$ are identical.

**Axiom 3.** $m$ is upper semi-continuous with respect to the distribution metric. That is to say, for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, \alpha, \phi)$, such that if $\| (\alpha, \phi), (\beta, \psi) \|_\ast < \delta$ then $m(\beta, \psi) < m(\alpha, \phi) + \varepsilon$. 
This last condition tells us that if the two measures \((f^{\alpha,\phi})^*(\mu)\) and 
\((f^{\beta,\psi})^*(\nu)\) are the same, then \(m(\alpha, \phi) = m(\beta, \psi)\). Hence the value \(m\)
is well defined on those measures on \(G^G\) which arise as such an image, and we can write

\[
m(\alpha, \phi) = m((f^{\alpha,\phi})^*(\mu)).
\]

Later on we will find ourselves in the situation where the rearrangement \((\alpha, \phi)\) is well-defined pointwise and the measure \(\mu\) is allowed to vary. In this case we will be more specific and write \(m_\mu(\alpha, \phi)\) or 
\(m_{\alpha,\mu}(\phi_1, \phi_2)\).

**Lemma 2.2.1.** Let \(m\) be a size. The identity map

\[
(\Gamma, \|\cdot\|_1) \rightarrow (\Gamma, m_\alpha)
\]
is uniformly continuous.

**Proof.** Let \(\phi_1, \phi_2 \in \Gamma\). As

\[
\|\phi_1, \phi_2\|_m \geq \mu(\{ x : \phi_1(x) \neq \phi_2(x) \})
\]

\[
= \|(\alpha\phi_1, \phi_1^{-1}\phi_2), \alpha\phi_1, \text{id}\|_1,
\]

Lemma 2.1.8 tells us that for any \(\delta_1 > 0\), there exists \(\delta > 0\), such that if 
\(\|\phi_1, \phi_2\|_m < \delta\) then 
\(\|(\alpha\phi_1, \phi_1^{-1}\phi_2), \alpha\phi_1, \text{id}\|_1 < \delta_1\).

Fix an arrangement \(\alpha_0\) and \(\phi_0 = \text{id}\). Let \(\varepsilon > 0\). Select \(\delta_1 = 
\delta(\varepsilon, \alpha_0, \text{id})\) from Axiom 3. It follows that if 
\(\|\phi_1, \phi_2\|_m < \delta\) then

\[
\|(\alpha\phi_1, \phi_1^{-1}\phi_2), \alpha\phi_1, \text{id}\|_1 < \delta_1,
\]

and hence

\[
m_\alpha(\phi_1, \phi_2) = m(\alpha\phi_1, \phi_1^{-1}\phi_2) < \varepsilon.
\]

\[\square\]

**Definition 2.2.2.** Let \(\alpha \in \mathcal{A}\), \(\{\phi_i\} \subseteq \Gamma\), and \(m\) a size. Define \(\Gamma_\alpha\) to be the equivalence classes of elements of \(\Gamma\) at an \(m_\alpha\) distance zero.

Thus \((\Gamma_\alpha, m_\alpha)\) is a metric space and we let \((\hat{\Gamma}_\alpha, m_\alpha)\) be its completion.

That is to say, \(\hat{\Gamma}_\alpha\) consists of sequences \(\{\phi_i\}\) which are \(m_\alpha\)-Cauchy, modulo the equivalence relation \(\{\phi_i\} \sim \{\psi_i\}\) if

\[
m_\alpha(\phi_i, \psi_i) \rightarrow 0.
\]

We write the elements of \(\hat{\Gamma}_\alpha\) as \(\langle \phi_i \rangle_\alpha\).

**Lemma 2.2.3.** For a fixed arrangement \(\alpha\) and size \(m\) the map \(\phi \rightarrow \alpha\phi\) from \(\Gamma\) to \(\mathcal{A}\) is well-defined as a map \(\Gamma_\alpha \rightarrow \mathcal{A}\) and extends to a uniformly continuous map from \((\hat{\Gamma}_\alpha, m_\alpha)\) to \((\mathcal{A}, \|\cdot\|_1)\). We refer to this map as \(P_{m,\alpha}(\langle \phi_i \rangle_\alpha) = \lim_i \alpha\phi_i\).
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**Proof.** As the map \( \phi \to \alpha \phi \) is uniformly continuous \( \Gamma_\alpha \to A \) it extends to the completion of \( \Gamma_\alpha \) (remember, \( A \) in the \( \|\cdot\|_1 \)-metric is complete.) \( \square \)

Thus if \( \langle \phi_i \rangle_\alpha \in \hat{\Gamma}_\alpha \) then \( \alpha \phi_i \to \beta \) for some \( \beta \) in \( A \). Hence we will at times abbreviate this element \( \langle \phi_i \rangle_\alpha \in \hat{\Gamma}_\alpha \) as \( \hat{\beta} \) indicating that it is a lift to \( \hat{\Gamma}_\alpha \) of the arrangement \( \beta \).

The arrangements in the range of \( P_{m,\alpha} \) are a first cut toward the \( m \)-equivalence class of \( \alpha \). The subset which is the actual equivalence class may be somewhat smaller.

Next, we see that since \( m \) satisfies Axiom 3, right multiplication in the full group \( \Gamma \) is an isometry. Specifically, we have the following lemma.

**Lemma 2.2.4.** Let \( \alpha \) be any arrangement. For all \( \phi_0 \in \Gamma \), the map \( \phi \to \phi \phi_0 \) is an \( m_\alpha \)-isometry, that is to say:

\[
m_\alpha(\phi_1 \phi_0, \phi_2 \phi_0) = m_\alpha(\phi_1, \phi_2).
\]

**Proof.** For all \( \phi_0, \phi_1, \phi_2 \in \Gamma \), we have that

\[
\|((\alpha \phi_0, \phi_0^{-1} \phi_1 \phi_0), (\alpha, \phi_1))\|_s = 0.
\]

Thus, by Axiom 3, for any \( \phi_0, \phi_1, \phi_2 \in \Gamma \),

\[
m(\alpha \phi_1 \phi_0, \phi_0^{-1} \phi_1 \phi_0) = m(\alpha \phi_1, \phi_1^{-1} \phi_2).
\]

That is to say, \( m_\alpha(\phi_1 \phi_0, \phi_2 \phi_0) = m_\alpha(\phi_1, \phi_2) \). \( \square \)

It is not necessarily the case that left multiplication is continuous. In fact, left multiplication need not preserve the equivalence classes of the pseudometric \( m_\alpha \). For example, consider the size

\[
m(\alpha, \phi) = \|\alpha, \alpha \phi\|_w.
\]

For \( G = \mathbb{Z} \) and \( \phi_1 = T_{1,0} \), we have that \( m^0(\text{id}, \phi_1) = 0 \), but \( m^0(\phi, \phi_1) = m^0(\text{id}, \phi \phi_1 \phi^{-1}) \). Here one can prove that \( m^0(\phi, \psi) = 0 \) if and only if \( \psi = T_{k,0} \). So if \( m^0(\phi, \phi_1) = 0 \), then \( \phi \phi_1 \phi^{-1} = T_{k,0} \). Thus, (using ergodicity and freeness), \( k = 1 \) and \( \phi \) commutes with \( T_{1,0} \).

Hence \( \phi \) is itself \( T_{n,0} \) for some \( n \), yet the full group is much larger than this.

We do have something akin to semi-continuity of right multiplication. Suppose \( \|\alpha \phi_i / \beta\|_w \to 0 \). Then, for a.e. \( x \), we have that

\[
\alpha(\phi_i \phi_1(x), \phi_i \phi_2(x)) \to \beta(\phi_1(x), \phi_2(x)),
\]

and so \( (\alpha \phi_i \phi_1, (\phi_i \phi_1)^{-1}(\phi_i \phi_2)) = (\alpha \phi_i \phi_1, \phi_1^{-1} \phi_2) \) converges in distribution to \( (\beta \phi_1, \phi_1^{-1} \phi_2) \). Axiom 3 now tells us that
\[ (*) \quad \lim_{i \to \infty} m_\alpha(\phi_1, \phi_2, \phi_1, \phi_2) \leq m_\beta(\phi_1, \phi_2). \]

**Definition 2.2.5.** We say that \( m \) is a \( 3^+ \) size if whenever \( \{ \phi_i \} \) is \( m_\alpha \)-Cauchy and hence \( \| \alpha \phi_i, \beta \|_1 \rightarrow 0 \), the inequality \((*)\) is an equality. In particular, this will be true if Axiom 3 is replaced by the following:

**Axiom 3**. Let \( \alpha \) be any ordering, \( \phi \in F \mathcal{G}(\mathcal{O}) \) and \( \varepsilon > 0 \). There exists \( \delta \) so that if \( \| (\alpha, \phi), (\beta, \psi) \|_s < \delta \), then \( | m(\alpha, \phi) - m(\beta, \psi) | < \varepsilon \); i.e. \( m \) is continuous in the distribution metric.

The isometry of right multiplication on \( \Gamma \) (Lemma 2.2.4), implies that the action of \( \Gamma \) by right multiplication extends as an isometric action of \( \Gamma \) on \( (\hat{\Gamma}_\alpha, m_\alpha) \). This action is clearly topologically transitive, since \( \hat{\Gamma} \) is an orbit closure.

In particular, we get the following

**Lemma 2.2.6.** Let \( \alpha \) be an arrangement and \( m \) a size. For any \( \langle \phi_i \rangle_\alpha \) in \( (\hat{\Gamma}_\alpha, m_\alpha) \),

\[ \lim_{j \to \infty} m_\alpha(\langle \phi_i \rangle_\alpha \phi_j^{-1}, (\text{id})_\alpha) = 0. \]

**Proof.** Let \( \{ \phi_i \} \) be an \( m_\alpha \)-Cauchy sequence, (representing the class \( \langle \phi_i \rangle_\alpha \)).

Fix \( j \). Compute that

\[ m_\alpha(\langle \phi_i \rangle_\alpha \phi_j^{-1}, (\text{id})_\alpha) = m_\alpha(\langle \phi_i \phi_j^{-1} \rangle_\alpha, (\text{id})_\alpha) \]
\[ = \lim_{i \to \infty} m_\alpha(\phi_i \phi_j^{-1}, \text{id}) \]
\[ = \lim_{i \to \infty} m_\alpha(\phi_i, \phi_j), \]

by Lemma 2.2.4.

Since \( \{ \phi_i \} \) is \( m_\alpha \)-Cauchy, for any \( \varepsilon > 0 \), there exists \( I \) such that for all \( i, j \geq I, m_\alpha(\phi_i, \phi_j) < \varepsilon \). Thus, for all \( j \geq I, \)

\[ \lim_{i \to \infty} m_\alpha(\phi_i, \phi_j) < \varepsilon. \]

The result follows.

In particular, this tells us that the action of the full group \( \Gamma \) on \( (\hat{\Gamma}_\alpha, m_\alpha) \) is minimal, i.e. every orbit is dense.

For a size \( m \), we would like to define a notion of \( m \)-equivalence between two arrangements \( \alpha \) and \( \beta \). Within the context of rearrangements, a natural candidate would be to say that two arrangements \( \alpha \) and \( \beta \) are \( m \)-equivalent if \( \beta \) is in the range of \( P_{m, \alpha} \). That is, there is a sequence of rearrangements \( (\alpha, \phi_i) \) with \( \alpha \phi_i \rightarrow \beta \) in \( L^1 \), such that
\{\phi_i\} is \(m_\alpha\)-Cauchy. The problem with this definition is that on the face of it the “equivalence relation” is not symmetric. More precisely, it is not clear that, in general, the \(m_\alpha\)-Cauchiness of \(\{\phi_i\}\) will imply \(m_\beta\)-Cauchiness of \(\{\phi_i^{-1}\}\).

The following theorem describes this situation more precisely:

**Theorem 2.2.7.** Suppose \(\alpha\) is an arrangement, \(m\) is a size, \(\{\phi_i\}\) is \(m_\alpha\)-Cauchy and \(\alpha\phi_i \to \beta\) in \(L^1\). The map

\[
P : (\Gamma, m_\beta) \to (\hat{\Gamma}_\alpha, m_\alpha),
\]

given by

\[
P(\phi) = \langle \phi, \phi \rangle_\alpha
\]
is a \(\Gamma\)-equivariant contraction, so that

\[
m_\alpha(P(\phi_1), P(\phi_2)) \leq m_\beta(\phi_1, \phi_2).
\]

Hence \(P\) extends to a \(\Gamma\)-equivariant contraction

\[
P : (\hat{\Gamma}_\beta, m_\beta) \to (\hat{\Gamma}_\alpha, m_\alpha).
\]

The following are equivalent:

1. \(\text{id} \in \text{Range}(P)\)
2. \(P\) is onto
3. \(P\) is an isometry
4. \(\{\phi_i^{-1}\}\) is \(m_\beta\)-Cauchy.

Lastly, if \(m\) is a \(3^+\) size, then for all \(\beta\) in the range of \(P_{m, \alpha}\), \(P\) is an isometry.

**Proof.** That \(P\) is a \(\Gamma\)-equivariant contraction, we verified earlier (in \(\ast\)) as a consequence of Axiom 3. This certainly implies that \(P\) extends to a \(\Gamma\)-equivariant contraction

\[
P : (\hat{\Gamma}_\beta, m_\beta) \to (\hat{\Gamma}_\alpha, m_\alpha).
\]

Moving on to the four equivalent statements: If \(\text{id} \in \text{Range}(P)\) then there exists \(\hat{\gamma} \in (\hat{\Gamma}_\beta, m_\beta)\) such that \(P(\hat{\gamma}) = \text{id}\). Suppose \(\hat{\gamma} = \langle \psi_i \rangle_\beta\). To show that \(P\) is onto, we need only show that \(P\) maps onto \(\Gamma\).

Let \(\phi \in \Gamma\). We'll show that \(P(\hat{\gamma} \phi) = \phi\).

In fact, \(P(\hat{\gamma} \phi) = \langle P(\psi_i \phi) \rangle_\alpha\). We need only prove that

\[
\lim_{i \to \infty} P(\psi_i \phi) = \phi.
\]
Compute that
\[
\lim_{i \to \infty} m_\alpha(P(\psi_i \phi), \phi) = \lim_{i \to \infty} m_\alpha((\{ \phi_j \psi_i \phi \}_j)_\alpha, \phi)
\]
\[
= \lim_{i \to \infty} \lim_{j \to \infty} m_\alpha(\phi_j \psi_i \phi, \phi)
\]
\[
= \lim_{i \to \infty} \lim_{j \to \infty} m_\alpha(\phi_j \psi_i, \text{id})
\]
since $\Gamma$ acts isometrically,
\[
= \lim_{i \to \infty} m_\alpha((\{ \phi_j \}_j)_\alpha, \text{id})
\]
\[
= \lim_{i \to \infty} m_\alpha((\{ \phi_j \}_j)_\alpha \psi, \text{id})
\]
\[
= \lim_{i \to \infty} m_\alpha(P(\psi_i), \text{id})
\]
\[
= 0,
\]
since $P(\hat{\gamma}) = \text{id}$.

Hence if $\text{id} \in \text{Range}(P)$ then $\Gamma \subseteq \text{Range}(P)$ and $P$ must be onto. Thus (1) and (2) are equivalent.

Next, we argue that if $\text{id} \in \text{Range}(P)$ then the sequence $\{ \phi_i^{-1} \}$ must be $m_\beta$-Cauchy.

Suppose $P(\hat{\gamma}) = \text{id}$, where $\hat{\gamma} = \langle \psi_i \rangle_\beta$. The fact that $P$ is an equivariant contraction implies that
\[
\langle \phi_i \rangle_\alpha = \langle \psi_i^{-1} \rangle_\alpha.
\]
To see this, simply compute that
\[
\lim_{i \to \infty} m_\alpha(\phi_i, \psi_i^{-1}) = \lim_{i \to \infty} m_\alpha(\phi_i \psi_i^{-1}, \text{id})
\]
\[
= \lim_{i \to \infty} m_\alpha((\phi_j)_\alpha \psi, \text{id})
\]
\[
= \lim_{i \to \infty} m_\alpha(P(\psi_i), P(\hat{\psi}))
\]
\[
\leq \lim_{i \to \infty} m_\beta(\psi_i, \hat{\psi}),
\]
\[
= 0.
\]

Define
\[
Q : (\Gamma, m_\alpha) \to (\Gamma_\beta, m_\beta),
\]
by
\[
Q(\phi) = \langle \psi \phi \rangle_\beta.
\]
Exactly as for $P$, argue that $Q$ is a $\Gamma$-equivariant contraction, so that
\[
\hat{m}_\beta(Q(a), Q(b)) \leq m_\alpha(a, b).
\]
Hence \( Q \) extends as a \( \Gamma \)-equivariant contraction

\[
Q : (\hat{\Gamma}, m_{\alpha}) \to (\hat{\Gamma}, m_{\beta}).
\]

Since \( \hat{\gamma} \in (\hat{\Gamma}, m_{\beta}) \), by Lemma 2.2.4, we see that

\[
Q(\langle \phi_{i}^{-1} \rangle_{\alpha}) = \text{id}.
\]

The above argument (applied to \( Q \)) now shows that

\[
\langle \phi_{i}^{-1} \rangle_{\beta} = \langle \psi_{i} \rangle_{\beta}.
\]

In particular this implies that \( \{ \phi_{i}^{-1} \} \) is \( m_{\beta} \)-Cauchy. Thus, since \( Q \) is a contraction, for \( \phi \in \Gamma \);

\[
m_{\beta}(Q(P(\phi)), \phi) = 0.
\]

Hence for all \( \hat{\phi} \in (\hat{\Gamma}_{\beta}, \hat{m}_{\beta}) \), we see that

\[
m_{\beta}(Q(P(\hat{\phi})), \hat{\phi}) = 0.
\]

Thus, for \( a, b \in (\hat{\Gamma}_{\beta}, \hat{m}_{\beta}) \),

\[
m_{\beta}(a, b) = m_{\beta}(Q(P(a)), Q(P(b)))
\]

\[
\leq m_{\alpha}(P(a), P(b))
\]

\[
\leq m_{\beta}(a, b),
\]

so that \( P \) is an isometry.

If \( P \) is an isometry, then of course, \( \{ \phi_{i}^{-1} \} \) is \( m_{\beta} \)-Cauchy, by Lemma 2.2.4. Hence, \( \text{id} \in \text{Range}(P) \). This completes the proof that statements (1) - (4) are equivalent.

If \( m \) is a \( 3^{+} \) size, then \( P \) is directly seen to be an isometry for all \( \beta \) in the range of \( P_{m,\alpha} \).

\[
\square
\]

**Definition 2.2.8.** There are three natural levels now on which to define \( m \)-equivalence classes. The first is the most functorial, as a subset of \( \hat{\Gamma}_{\alpha} \) we can set

\[
\hat{E}_{m}(\alpha) = \{ \langle \phi_{i} \rangle_{\alpha} \in \hat{\Gamma}_{\alpha} : \alpha \phi_{i} \to \beta \text{ and } \langle \phi_{i}^{-1} \rangle_{\beta} \in \hat{\Gamma}_{\beta} \}.
\]

We can also consider the relation on arrangements given by

\[
E_{m}(\alpha) = \{ \beta | \alpha \overset{m}{\sim} \beta \} = P_{m,\alpha}(\hat{E}_{m}(\alpha)).
\]

Thirdly of course we can consider the category of free and ergodic \( G \)-actions \( T \) and \( S \) and say \( S \) is \( m \)-equivalent to \( T \) if there is an arrangement \( \beta \in E_{m}(\alpha_{T}) \) with \( T^{\beta} \) conjugate to \( S \). We indicate all three of these relations by the symbol \( \overset{m}{\sim} \). Thus we will write \( \langle \phi_{i} \rangle_{\alpha} \overset{m}{\sim} \langle \psi_{i} \rangle_{\alpha} \), \( \alpha \overset{m}{\sim} \beta \) and \( T \overset{m}{\sim} S \).
2.2. Definition of a size and $m$-equivalence

We investigate the first two of these a bit. We will show that $\hat{E}_m(\alpha)$ is a dense $G_\delta$ subset of $\hat{\Gamma}_\alpha$ directly by exhibiting it as a countable intersection of open sets. We will also show that each equivalence class $E_m(\alpha)$ as a subset of the arrangements, can be endowed with a natural $m_\alpha$-metric making it a universal $G_\delta$. We obtain this latter result by showing the map $\hat{E}_m(\alpha) \to E_m(\alpha)$ is obtained by considering $\hat{E}_m(\alpha)$ modulo a natural group of isometries of $\hat{E}_m(\alpha)$.

Our first task is to see that on the level of arrangements $m_\alpha \sim$ is an equivalence relation. It then follows automatically for free and ergodic actions.

We begin by putting a natural metric on $E_m(\alpha)$. As $P_{m,\alpha}$ is continuous from $\hat{\Gamma}_\alpha \to \mathcal{A}$, for any $\beta$ in the range of $P_{m,\alpha}$, its pullback $P_{m,\alpha}^{-1}(\beta)$ will be a closed set. We can use the Hausdorff metric on these closed sets to put a metric on the equivalence class $E_m(\alpha)$. More precisely, for $\beta_1, \beta_2 \in E_m(\alpha)$, define

$$m(\beta_1, \beta_2) = \inf_{(\phi_i)_{\alpha} \in P_{m,\alpha}(\beta_1)} m_{\alpha}(\langle \phi_i \rangle_{\alpha}, \langle \psi_i \rangle_{\alpha}) = m_{\alpha}(P_{m,\alpha}^{-1}(\beta_1), P_{m,\alpha}^{-1}(\beta_2)).$$

At this point it is not quite evident that this is a metric and not just a pseudometric. We have the following trivial consequence to Axiom 2.

**Lemma 2.2.9.** For every $\varepsilon > 0$, there exists $\delta > 0$, such that if $m(\alpha, \beta) < \delta$ then $\|\alpha, \beta\|_1 < \varepsilon$. That is to say, the identity is uniformly continuous from $m$ to $\|\cdot, \cdot\|_1$. Hence $m(\alpha, \beta)$ is a metric.

In particular, $\alpha \sim^m \beta$ if the associated map $P$ defined in Theorem 2.2.7 is an isometry.

We can put this together now in a simple form:

**Lemma 2.2.10.** $\alpha \sim^m \beta$ if and only if there exists a $\Gamma$-equivariant isometry $P : (\hat{\Gamma}, \hat{m}_\alpha) \to (\hat{\Gamma}, \hat{m}_\beta)$, with $P_{m,\beta}P = P_{m,\alpha}$.

**Proof.** If $\alpha \sim^m \beta$ then the existence of such a $P$ follows from Theorem 2.2.7. Conversely, suppose such a $P$ exists. As $P_{m,\alpha}(\text{id}) = \alpha$, we see that $P_{m,\beta}(P(\text{id})) = \alpha$. Now $P(\text{id}) = \langle \phi_i^{-1} \rangle_{\beta} \in (\hat{\Gamma}_\beta, \hat{m}_\beta)$. Thus, $P_{m,\beta}(\langle \phi_i^{-1} \rangle_{\beta}) = \alpha$. 

\[\hat{\Gamma}_\alpha \xrightarrow{P} \hat{\Gamma}_\beta \]

\[\xymatrix{ \hat{\Gamma}_\alpha \ar@{.>}[r]^P \ar[d]_{P_{m,\alpha}} & \hat{\Gamma}_\beta \ar[d]^{P_{m,\beta}} \\
\mathcal{A} }\]
For any $j$, we compute that
\[ m_{\alpha}(P^{-1}(\text{id}), \phi_j) = m_{\beta}(P(P^{-1}(\text{id})), P(\phi_j)) \]
\[ = m_{\beta}(\text{id}, (\phi^{-1}_i)_{\beta, j}) \]
\[ \to 0 \text{ in } j, \text{ by Lemma 2.2.4.} \]

Thus $P^{-1}(\text{id}) = (\phi_j)_\alpha \in (\hat{\Gamma}_\alpha, m_\alpha)$, and $\text{id} \in \text{Range}(P)$. □

The following theorem is now evident.

**Theorem 2.2.11.** The relation $\sim_\alpha$ on $\mathcal{A}$ is an equivalence relation.

From this we see that $\sim_\alpha$ breaks $\mathcal{A}$ into disjoint equivalence classes on each of which we have defined a metric. Our final is to see that relative to this metric each of these classes is a Polish space. We begin with the classes $\hat{E}_m(\alpha)$.

**Theorem 2.2.12.** The set $\hat{E}_m(\alpha)$ is a $G_\delta$-subset of $\hat{\Gamma}_\alpha$. As the full group is $m_\alpha$-separable the $m_\alpha$-topology on $\hat{E}_m(\alpha)$ is Polish.

**Proof.** For any $\phi$ and $\psi$ in the full group and $\varepsilon > 0$ let
\[ \mathcal{O}(\phi, \psi, \varepsilon) = \{(\phi_i)_\alpha \in \hat{\Gamma}_\alpha : \alpha \phi_i \to \beta \text{ and } m_{\beta}(\phi, \psi) < m_\alpha((\phi_i)_\alpha \phi_i, (\phi_i)_\alpha \psi) + \varepsilon \}. \]

Notice that

1. $\hat{E}_m(\alpha) \subseteq \mathcal{O}(\phi, \psi, \varepsilon)$ for all $\phi, \psi$, and $\varepsilon$ and

2. if $(\phi_i)_\alpha \in \mathcal{O}(\phi, \psi, \varepsilon)$ for all $\phi, \psi$, and $\varepsilon$ then
\[ m_{\beta}(\phi, \psi) \leq m_\alpha(P(\phi), P(\psi)). \]

As we always have $m_{\beta}(\phi, \psi) \geq m_\alpha(P(\phi), P(\psi))$, we conclude $P$ is an isometry and hence $(\phi_i)_\alpha \in \hat{E}_\alpha$.

The full group is separable in the $L^1$-topology and so by Axiom 2, is separable in the $m_\alpha$-topology and we can find a countable collection $\{\phi_i\}$ dense in all the $m_\alpha$ or $m_\beta$-topologies. Hence
\[ \cap_{i,j,k} \mathcal{O}(\phi_i, \phi_j, 1/k) = \hat{E}_m(\alpha). \]

It remains to see that the sets $\mathcal{O}(\phi, \psi, \varepsilon)$ are open in $\hat{\Gamma}_\alpha$. Suppose $(\phi_i)_\alpha \in \mathcal{O}(\phi, \psi, \varepsilon)$ and hence there is an $\varepsilon > 0$ with
\[ m_{\beta}(\phi, \psi) < m_\alpha((\phi_i)_\alpha \phi_i, (\phi_i)_\alpha \psi) + \varepsilon + \varepsilon. \]
By Axiom 2, Lemma 2.1.8, and Axiom 3 there is a $\delta_0 > 0$ so that if
\[ m_\alpha(\langle \phi_i \rangle_\alpha, \langle \phi_i' \rangle_\alpha) < \delta_0 \]
then $\| \beta, \beta' \|_1$ will be sufficiently small to imply that
\[ m_{\beta'}(\phi, \psi) < m_\beta(\phi, \psi) - \epsilon/3. \]
As
\[
m_\alpha(\langle \phi_i \rangle_\alpha \phi, \langle \phi_i \rangle_\alpha \psi) \\
\leq m_\alpha(\langle \phi_i' \rangle_\alpha \phi, \langle \phi_i' \rangle_\alpha \psi) + m_\alpha(\langle \phi_i \rangle_\alpha \phi, \langle \phi_i' \rangle_\alpha \psi) \\
+ m_\alpha(\langle \phi_i \rangle_\alpha \psi, \langle \phi_i' \rangle_\alpha \psi) \\
= m_\alpha(\langle \phi_i' \rangle_\alpha \phi, \langle \phi_i' \rangle_\alpha \psi) + 2m_\alpha(\langle \phi_i \rangle_\alpha \phi, \langle \phi_i' \rangle_\alpha \psi),
\]
making sure that $\delta_0 < \epsilon/3$ we will have
\[
m_{\beta'}(\phi, \psi) < m_\beta(\phi, \psi) - \epsilon/3 \\
< m_\alpha(\langle \phi_i \rangle_\alpha \phi, \langle \phi_i \rangle_\alpha \psi) + \epsilon + 2\epsilon/3 \\
< m_\alpha(\langle \phi_i' \rangle_\alpha \phi, \langle \phi_i' \rangle_\alpha \psi) + \epsilon
\]
and $\beta' \in \mathcal{O}(\phi, \psi, \epsilon)$. \hfill \Box

Suppose that $\hat{\beta}_1 = \langle \phi_i^1 \rangle_\alpha$ and $\hat{\beta}_2 = \langle \phi_i^2 \rangle_\alpha$ are in $\hat{E}_m(\alpha)$ with $\alpha \phi_i^1 \to \beta$ and $\alpha \phi_i^2 \to \beta$. That is to say, $P_{m,\alpha}(\hat{\beta}_1) = P_{m,\alpha}(\hat{\beta}_2) = \beta$. This means that for all $\phi$ and $\psi$ that
\[
m_{\alpha}(\hat{\beta}_1 \phi, \hat{\beta}_1 \psi) = m_{\beta}(\phi, \psi) = m_{\alpha}(\hat{\beta}_2 \phi, \hat{\beta}_2 \psi).
\]
Beginning the definition of an $m_\alpha$-isometry $I_{\hat{\beta}_1, \hat{\beta}_2}$ by setting
\[
I_{\hat{\beta}_1, \hat{\beta}_2}(\hat{\beta}_1 \phi) = \hat{\beta}_2 \phi.
\]
The above calculation implies that $I_{\hat{\beta}_1, \hat{\beta}_2}$ is an $m_\alpha$-isometry where it is defined, and, as the $\hat{\beta}_1 \phi$ are dense in $\hat{\Gamma}_\alpha$, $I_{\hat{\beta}_1, \hat{\beta}_2}$ will extend to an isometry of $\hat{\Gamma}_\alpha$. Notice that this makes $I_{\hat{\beta}_1, \hat{\beta}_2}$ commute with the action of the full group on $\hat{\Gamma}_\alpha$.

**Lemma 2.2.13.** For all $\hat{\beta}_1$, $\hat{\beta}_2$ in $\hat{E}_m(\alpha)$ with $P_{m,\alpha}(\hat{\beta}_1) = P_{m,\alpha}(\hat{\beta}_2)$, we have
\[
P_{m,\alpha}I_{\hat{\beta}_1, \hat{\beta}_2} = P_{m,\alpha}.
\]
**Proof.** As $P_{m,\alpha}$ is equivariant with the action of the full group,
\[
(P_{m,\alpha}I_{\hat{\beta}_1, \hat{\beta}_2}(\hat{\beta}_1 \phi) = P_{m,\alpha}(I_{\hat{\beta}_1, \hat{\beta}_2}(\hat{\beta}_1 \phi)) = \beta \phi = P_{m,\alpha}(\hat{\beta}_1 \phi).
\]
This now extends to all of $E_m(\alpha)$ as the $\hat{\beta}_1 \phi$ are dense. \hfill \Box

**Definition 2.2.14.** Let $\mathcal{I}$ consist of those $m_\alpha$-isometries of $\hat{\Gamma}_\alpha$ commuting with the action of the full group and satisfying

$$P_{m_\alpha} I = P_{m_\alpha}.$$

This is a complete and separable metrizable space under pointwise convergence. Notice then that for any $\hat{\beta}_1 \in \hat{E}_m(\alpha)$ and $I \in \mathcal{I}$, setting $\hat{\beta}_2 = I(\hat{\beta}_1)$, we will have $\hat{\beta}_2 \in \hat{E}_m(\alpha)$ and

$$I = I_{\hat{\beta}_1, \hat{\beta}_2}.$$ 

Furthermore, as right multiplication by elements of the full group is an $m_\alpha$-isometry of $\hat{\Gamma}_\alpha$,

$$m_\alpha(\hat{\beta}, I(\hat{\beta})) = C(\hat{\beta})$$

is a constant on $\hat{\Gamma}_\alpha$.

We now argue that the orbits of $\mathcal{I}$ are closed sets. Suppose $I_i(\hat{\beta}_0)$ converges to some $\hat{\beta}_1$. Then in particular the $I_i(\hat{\beta}_0)$ will be $m_\alpha$-Cauchy. But the above remark implies that $I_i(\hat{\beta})$ is $m_\alpha$-Cauchy for all $\hat{\beta}$, in particular converges to some $I(\hat{\beta}) \in \hat{\Gamma}_\alpha$. That is to say, $I_i \rightarrow I$ uniformly and we conclude that $I \in \mathcal{I}$ and all orbits are closed.

**Lemma 2.2.15.** Using the infimum metric, the space $\hat{\Gamma}_\alpha/\mathcal{I}$ is a complete separable metric space.

**Proof.** To see that the infimum is a metric suppose

$$\inf_{I_1, I_2 \in \mathcal{I}} (m_\alpha(I_1(\hat{\beta}_1), I_2(\hat{\beta}_2))) = 0$$

then of course

$$\inf_{I \in \mathcal{I}} (m_\alpha(\hat{\beta}_1, I(\hat{\beta}_2))) = 0$$

and there will be a sequence $I_i(\hat{\beta}_1) \rightarrow \hat{\beta}_2$. But this says $I_i(\hat{\beta}_1)$ is $m_\alpha$-Cauchy, and hence $I_i(\hat{\beta})$ is $m_\alpha$-Cauchy for all $\hat{\beta}$. That is to say, the $I_i$ converge uniformly to another $I \in \mathcal{I}$ implying $\hat{\beta}_2 = I(\hat{\beta}_1)$.

Thus the infimum is a metric on $\hat{\Gamma}_\alpha/\mathcal{I}$.

Separability follows from the fact that $\hat{\Gamma}_\alpha$ is separable. \hfill \Box

We now show that $E_m(\alpha)$ is also a $G_\delta$-subset of $\mathcal{A}$ by showing that it imbeds as $\hat{E}_m(\alpha)/\mathcal{I}$ which we show to be a residual subset of $\hat{\Gamma}_\alpha/\mathcal{I}$. We achieve the embedding by lifting $\beta \in E_m(\alpha)$ to $P_{m_\alpha}^{-1}(\beta) \cap \hat{E}_m(\alpha)$ which we note maps to a singleton in $\hat{\Gamma}_\alpha/\mathcal{I}$. It follows directly from the
definitions that this is a continuous embedding. This implies \( E_m(\alpha) \) is a universal \( G_\delta \), that is to say is a \( G_\delta \)-subset of any metric space in which it is embedded.

**Lemma 2.2.16.** The sets \( \mathcal{O}(\phi, \psi, \varepsilon) \) of Theorem 2.2.12 are \( \mathcal{I} \)-invariant. Hence

\[
\hat{E}_m(\alpha)/\mathcal{I} = \bigcap_{i,j,k} \mathcal{O}(\phi_i, \phi_j, 1/k)/\mathcal{I}
\]
a \( G_\delta \)-subset.

**Proof.** Remember that

\[
\mathcal{O}(\phi, \psi, \varepsilon) = \{ \hat{\beta} : P_{m,\alpha}(\hat{\beta}) = \beta \text{ and } m_\beta(\phi, \psi) < m_\alpha(\hat{\beta}\phi, \hat{\beta}\psi) + \varepsilon \}.
\]

For \( I \in \mathcal{I} \) we have both \( P_{m,\alpha}I = P_{m,\alpha} \) and \( I \) is an \( m_\alpha \)-isometry. These combine to say that if \( \hat{\beta}' = I(\hat{\beta}) \) then \( P_{m,\alpha}(\hat{\beta}') = \hat{\beta} \) and

\[
m_\alpha(\hat{\beta}'\phi, \hat{\beta}'\psi) = m_\alpha(I(\hat{\beta}\phi), I(\hat{\beta}\psi)) = m_\alpha(\hat{\beta}\phi, \hat{\beta}\psi).
\]

It is now clear that

\[
I(\mathcal{O}(\phi, \psi, \varepsilon)) = \mathcal{O}(\phi, \psi, \varepsilon).
\]

It is obvious that if \( \mathcal{O} \) is an \( \mathcal{I} \) invariant open set then \( \mathcal{O}/\mathcal{I} \) is open. It is also obvious that the collection of \( \mathcal{I} \)-invariant sets form a Boolean algebra, with moding out by \( \mathcal{I} \) equivariant with the Boolean operations. This completes the result. \( \square \)

Perhaps the best heuristic to take away from this section is the image of the \( m \)-equivalence classes as foliating the set of arrangements. Each leaf of this foliation is metrized by its \( m_\alpha \) as a Polish space and \( \Gamma \) acts on each leaf minimaly and isometrically.

### 2.3. Seven Examples

Having developed the axiomatics of \( m \)-equivalence we now give a list of examples to indicate the range of equivalence relations that can be brought under this perspective. In [36] a number of examples and classes of examples are discussed. Some of those are quite speculative. The appendix of this work demonstrates how to bring all those examples under the umbrella we open here. The examples we discuss in this section are those which are most obviously significant and directly related to classical issues in ergodic theory. As part of this discussion we give some general principles that underly many of these examples as the beginning of we expect a fruitful study of what a size might look like in general.

Many examples of sizes have the common feature of being integrals of some pointwise calculation of the distortion of a single orbit. To
make this precise we first review some material about bijections of $G$. Remember that $\mathcal{B}$ is the space of all bijections of the group $G$ with the product topology, $\mathcal{G}$ is the space of bijections fixing id and we metrized both with a complete metric $d$. The group $G$ can be regarded as a subgroup of $\mathcal{B}$ acting by left multiplication, $(g(g') = gg')$. The map $\hat{H} : \mathcal{B} \to \mathcal{G}$ given by $\hat{H}(q) = qq(id)^{-1}$ is a contraction in $d$. Also $G$ acting by right multiplication conjugates $\mathcal{B}$ to itself giving an action of $G$ on $\mathcal{B}$. ($T_g(q)(g') = q(g'g)g^{-1}$.) We view this action by representing an element $q \in \mathcal{B}$ by a map $f : G \to G$, $f(g) = q(g)g^{-1}$. Those maps $f \in G^G$ that arise from bijections are a $G_\delta$ and hence a Polish space we call $F$. The map $q \to f$ is obviously a homeomorphism from $\mathcal{B}$ to $F$. For $f \in F$ let $Q(f)$ be the associated bijection and for $q \in \mathcal{B}$ let $F(q)$ be the associated name in $G^G$. The action of $G$ on $\mathcal{B}$ in its representation as $F$ is the shift action $\sigma_g(f)(g') = f(g'g)$. Any rearrangement pair $(\alpha, \phi)$ then gives rise to an ergodic shift invariant measure on this Polish subset of $G^G$ and any ergodic shift invariant measure is an ergodic action of $G$ with a canonical rearrangement pair. The probability measures on a Polish space are weak* Polish [49] and hence the invariant and ergodic measures on this Polish space are weak* Polish.

We will now define a general class of sizes that arise as integrals of valuations made on the bijections $q_{x, \phi}^\alpha$.

**Definition 2.3.1.** A Borel $D : \mathcal{B} \to \mathbb{R}^+$ is called a size kernel if it satisfies:

1. $D(q) \geq 0$.
2. $D(id) = 0$.
3. $D(q(id)^{-1}q^{-1}q(id)) = D(q)$.
4. $D(q_1(id)q_2q_1^{-1}(id)q_1) \leq D(q_1) + D(q_2)$.
5. For every $\varepsilon > 0$ there is a $\delta > 0$ so that if $D(q) < \delta$ then $d(id, H(q)) < \varepsilon$.
6. The function $\mu \to \int D(q(f)) \, d\mu$ is weak* continuous on space of shift invariant measures $\mu$ on the Polish space $F$.

Note: an element of $G$ regarded as an element of $\mathcal{B}$ acts by left multiplication.

The complex form of conditions 3) and 4) arise from the following considerations. When an orbit is viewed as a copy of $G$ the base point $x$ sits on the identity element of the group. When acted on by some rearrangement the identity moves, i.e. the point $x$ now is based at a different point in $G$. Hence it is necessary to view both $q^{-1}$ and $q_2$ as based at this new origin when they act. Writing it out explicitly on
an orbit we have the identities

\[ q_x^{\alpha, \psi_2 \circ \psi_1} = q_x^{\alpha, \psi_1}(\text{id})q_x^{\alpha, \psi_2}(q_x^{\alpha, \psi_1}(\text{id}))^{-1}q_x^{\alpha, \psi_1} \]

and

\[ q_x^{\alpha, \psi_{i=1}} = q_x^{\alpha, \psi}(\text{id})(q_x^{\alpha, \psi})^{-1}q_x^{\alpha, \psi}(\text{id}). \]

and now conditions 3) and 4) become

\[ D(q_x^{\alpha, \psi_2 \circ \psi_1}) \leq D(q_x^{\alpha, \psi_1}) + D(q_x^{\alpha, \psi_1 \circ \psi_2}) \]

and

\[ D(q_x^{\alpha, \psi}) = D(q_x^{\alpha, \psi \circ \psi_1}) \]

For size kernels \( D \) defined solely in terms of \( H(q) \) these two become even simpler as \( H(q_1(\text{id})q_2q_1^{-1}(\text{id})q_1) = H(q_2)H(q_1) \) and \( H(q(\text{id})^{-1}q^{-1}q(\text{id})) = H(q)^{-1} \).

For a size kernel \( D \) we define

\[ m^D(\alpha, \phi) = \int D(q_x^{\alpha, \phi}) \, d\mu(x). \]

We call such an \( m^D \) an **integral size**.

In condition 6) on \( D \) we could have asked for only upper semi continuity and still obtained that \( m^D \) was a size. All examples though are continuous here so we ask for the stronger condition and obtain a stronger conclusion:

**Theorem 2.3.2.** For \( D \) a size kernel, \( m^D \) is a \( 3^+ \) size.

**Proof.** First note that
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\[ q_x^{\alpha \phi_1, \phi_2}(g) = \alpha(\phi_1(x), \phi_1 \phi_2(T^\alpha \phi_1(x))) \]
\[ = \alpha(\phi_1(x), \phi_1 \phi_2(\phi_1^{-1} T^\alpha \phi_1(x))) \]
\[ = q_{\phi_1(x)}^{\alpha \phi_1, \phi_2, \phi_1^{-1}}(g) \]

\[ m^D_\alpha(\phi_1, \phi_3) = \int D(q_x^{\alpha \phi_1, \phi_1^{-1} \phi_3}) \, d\mu = \int D(q_{\phi_1(x)}^{\alpha \phi_3 \phi_1^{-1}}) \, d\mu = \int D(q_{\phi_1(x)}^{\alpha \phi_3 \phi_2 \phi_1^{-1} \phi_3}) \, d\mu \]

and from the above identity and condition 3,
\[ \leq \int D(q_{\phi_1(x)}^{\alpha \phi_3 \phi_2 \phi_1^{-1}}) + D(q_{\phi_1(x)}^{\alpha \phi_3 \phi_2 \phi_1^{-1}}) \, d\mu = \int D(q_{\phi_1(x)}^{\alpha \phi_3 \phi_2 \phi_1^{-1}}) \, d\mu + \int D(q_x^{\alpha \phi_1 \phi_1^{-1} \phi_2}) \, d\mu = m^D_\alpha(\phi_1, \phi_2) + m^D_\alpha(\phi_2, \phi_3). \]

Condition 3. gives symmetry as
\[ D(q_x^{\alpha \phi_1, \phi_2 \phi_1^{-1}}) = D(q_x^{\alpha \phi_2, \phi_1 \phi_1^{-1}}) = m^D_\alpha(\phi_2, \phi_1). \]

and \( m^D_\alpha \) is a pseudometric on \( \Gamma \). Axiom II of a size follows directly from condition 5. Condition 6 is precisely that Axiom III should hold. \( \square \)

Example 1 (Conjugacy and Orbit Equivalence).

These first two examples are the extremes of what is possible. For one the equivalence class will be simply the full group orbit and for the other it will be the entire set of arrangements. Both of the pseudometrics \( d(q, \text{id}) \) and \( d(H(q), \text{id}) \) are easily seen to be size kernels and so both
\[ m^1(\alpha, \phi) = \|(\alpha, \phi), (\alpha, \text{id})\|_{\alpha}^1 \]
\[ m^0(\alpha, \phi) = \|(\alpha, \phi), (\alpha, \text{id})\|_{\alpha}^0 \]
are \( 3^+ \) sizes.

As \( d \) makes \( B \) complete, relative to \( m^1 \) a class of sequences \( \langle \phi_i \rangle_\alpha \in \hat{\Gamma}_\alpha \) iff \( \phi_i \rightarrow \phi \) in probability. Thus \( \alpha \overset{m^1}{\sim} \beta \) iff \( \beta = \alpha \phi \) i.e. they differ by an element of the full group and the equivalence class of \( \alpha \) is exactly its full group orbit. Note in particular \( T^\alpha \) and \( T^\beta \) will be conjugate actions. To tie this relation into our work here notice, in Chapter 7 where we define the notion of an \( m \)-finitely determined action, this definition reduces to Ornstein’s classical characterization of the Bernoulli actions as the finitely determined actions for \( m^1 \).
As for $m^0$, for any $\alpha$ and $\beta$ one can use the Ornstein-Weiss Rohklin lemma (Lemma 2.1.5) to construct a sequence of $\phi_i$ with $\alpha \phi_i \to \beta$ in $L^1$ with the sequence $\phi_i$ an $m_\alpha$ Cauchy sequence. Thus all arrangements are $m^0$ equivalent. Dye’s Theorem [8] and the Theorem of Connes, Feldman and Weiss [1] now tell us that any two ergodic actions of $G$ are $m^0$ equivalent.

We tie this example into our work. First a reminder of the distribution topology. For $G$ a countable and amenable group and $\Sigma$ a finite labeling set, the space of probability measures on $\Sigma^G$ forms a compact metrizable space. For any measure preserving action of $G$ and $\Sigma$ valued partition the map from points to $\Sigma, G$-names will project the invariant measure to a measure on $\Sigma^G$ and will make a pseudometric space of such processes (pairs of actions and partitions). For two such pairs $(T, P)$ and $(S, Q)$ let $\|(T, P), (S, Q)\|_*$ be some metric giving this weak* or distribution pseudotopology. We state a lemma concerning ergodic actions of $G$:

**Lemma 2.3.3.** Let $G$ a countable and amenable group, $T^\alpha$ a free and ergodic action of $G$ on the standard space $(X, \mathcal{F}, \mu)$, and $P$ a finite $\Sigma$-valued partition of $X$. For each $\epsilon > 0$ there is a $\delta > 0$ so that for any other free and ergodic action $S^\beta$ of $G$ on $(Y, \mathcal{G}, \nu)$ and partition $Q : Y \to \Sigma$ satisfying:

1. $\|(T^\alpha, P), (S^\beta, Q)\|_* < \delta$

and every $\delta_1 < 0$ there is a $\phi$ in the full group of $\beta$ and a partition $Q'$ of $Y$ with

a. $m^0(\beta, \phi) < \epsilon$

b. $\nu(Q \Delta Q') < \epsilon$ and

1'. $\|(T^\alpha, P), (S^{\beta \phi}, Q')\|_* < \delta_1$

We leave the proof as an exercise for the reader. A version of this fact for Kakutani Equivalence is found in Lemma 4.3 of [54]. The reader can use this as an outline of to how to proceed. One concludes from this that all ergodic actions of $G$ are weakly $m^0$-finitely determined (see Definition 7.2.5) and as $m^0$ is a $3^+$ size, Theorem 7.2.6 now implies all ergodic actions are $m^0$-f.d. giving an alternate albeit elaborate proof of Dye’s theorem using our machinery.

Before we continue to other examples we make a few general observations concerning size kernels. First we can w.l.o.g. assume that all size kernels are bounded by 1 as replacing $D$ by the supremum of
$D$ and 1 will maintain the axioms and will not change the associated equivalence relation. Notice next that one evaluates the size $m^D$ of a rearrangement measure by calculating $\int D(Q(f)) \, d\mu(f)$ where $\mu$ is some shift invariant measure on $F$. Suppose $F_0 \subseteq F$ is a shift invariant set with $\mu(F_0) = 1$ for all shift invariant probability measures $\mu$. Assume as well that $F_0$ contains the identity and if it contains $F(q)$ then it also contains $F(q^{-1})$. Notice that changing $D$ outside of $F_0$ will have no effect on the evaluation of $m^D$. In particular if $D$ is initially only defined on $F_0$, is bounded by 1 there, and satisfies the axioms of a size kernel (where applicable), then if we set $D(Q(f)) = 1$ for $f \notin F_0$ we would extend $D$ so as to be a size kernel.

We now give an explicit example of such an $F_0$. Suppose $G$ is $\mathbb{Z}^n$ and $B_N = [-N, N]^n$ is the standard Følner sequence of boxes. For shift invariant measures on $G^G$ the pointwise ergodic theorem holds along this sequence $B_N$. For each $f \in F$ set $\Delta_M(f)$ to be the upper density of the set $\{\vec{x} | f(\vec{x}) \notin B_M\}$ calculated along the sequence of sets $B_N$ as $N \uparrow \infty$. Let $F_0$ consist of those $f$ for which $\lim_{M \to \infty} \Delta_M(f) = 0$. The pointwise ergodic theorem tells us this set has measure 1 for all shift invariant measures. Hence when working in $\mathbb{Z}^n$ one need only define a size kernel $D$ on such $f$. Notice that for such $f$ one will have

$$\lim_{N \to \infty} \frac{\# \{ \vec{x} \in B_N | Q(f)(\vec{x}) \notin B_N\}}{\# B_N} = 0.$$ 

We describe a class of examples that take advantage of these observations and this choice for an $F_0$.

**Example 2** (Kakutani Equivalence).

The development of Kakutani equivalence in $\mathbb{Z}^n$ can be found in [6] and a complete development of the equivalence theorem for it in [14]. What we present here is an approach that brings this example into our context. For this example let $G = \mathbb{Z}^n$ and $B_N = [-N, N]^n$ be the standard Følner sequence of boxes centered at $\vec{0}$. We begin with a metric on $\mathbb{Z}^n$ given by

$$\tau(\vec{u}, \vec{v}) = \min (\|\vec{u}/\|\vec{u}\| - (\vec{v}/\|\vec{v}\|)) + |\ln(\|\vec{v}\|) - \ln(\|\vec{u}\|)|, 1)$$

(assuming $\|\vec{0}/\|\vec{0}\| = \vec{0}$). What is important about $\tau$ are the following two properties:

1. $\tau$ is a metric on $\mathbb{Z}^n$ bounded by 1 and
2. $\vec{u}$ and $\vec{v}$ are $\tau$ close iff the norm of their difference is small in proportion to both of their norms.
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For \( h \in \mathcal{G} \) set \( B_N(h) = \{ \vec{v} \in B_N | h(\vec{v}) \in B_N \} \) (those elements of \( B_N \) mapped into \( B_N \) by \( h \)). Now set

\[
k(h) = \sup_N \left( \frac{1}{\# B_N} \left( \sum_{\vec{v} \in B_N(h)} \tau(\vec{v}, h(\vec{v})) + \# \{ \vec{v} \in B_N | h(\vec{v}) \notin B_N \} \right) \right).
\]

Now set \( K(q) = k(H(q)) \).

**Lemma 2.3.4.** The function \( K \) is a size kernel.

**Proof.** It is a calculation that \( k(h) = k(h^{-1}) \) as in fact this equality already holds for each \( N \). It is also true that \( k(h_2 \circ h_1) \leq k(h_2) + k(h_1) \) as it is true for each \( N \) before taking the \( \sup_N \). That \( K \) satisfies the first five conditions of a size kernel is now direct. We get 3) and 4) from the observation that \( K(q) \) only depends on \( H(q) \). As \( \tau \) is a metric for \( K(q) \) to be small \( H(q) \) must fix a large (finite of course) number of vectors \( \vec{v} \). To obtain 6) suppose \( \mu \) is some shift invariant measure on \( G^n \), hence supported on \( F_0 \). For \( q \in F_0 \) as \( N \to \infty \) we have \( |B_N(H(q))|/|B_N| \to 1 \). Moreover for each \( N \) this value is continuous and so its expected value relative to \( \mu \) is weak* continuous. For any fixed \( \vec{v} \), \( \{ q|H(q)(\text{id}) = \vec{v} \} \) is a clopen set and hence its measure is weak* continuous in \( \mu \). As \( \vec{v} \) varies these sets form a countable partition of \( G^n \) and so for any \( \varepsilon > 0 \) there is an \( N_0 \) and a neighborhood \( U \) of \( \mu \) so that for \( \nu \in U \) also invariant and \( N \geq N_0 \), letting \( h = h(q(f)) \),

\[
\int \frac{1}{\# B_N} \left( \sum_{\vec{v} \in B_N(h)} \tau(\vec{v}, h(\vec{v})) + \# \{ \vec{v} \in B_N | h(\vec{v}) \notin B_N \} \right) d\mu(f) < \varepsilon.
\]

For each \( N < N_0 \) the calculation

\[
\frac{1}{\# B_N} \left( \sum_{\vec{v} \in B_N(h)} \tau(\vec{v}, h(\vec{v})) + \# \{ \vec{v} \in B_N | h(\vec{v}) \notin B_N \} \right)
\]

is continuous and hence the supremum of these values for \( N < N_0 \) is continuous and so its integral is weak* continuous in \( \mu \). It follows that in some sub-neighborhood \( U' \subseteq U \) we will have a variation of at most \( \varepsilon \) in the value \( \int K(q(f)) d\nu \).

The use of \( \tau \) here is not the usual calculation taken to construct Kakutani equivalence but noting that for \( \tau \) to be small simply means the distance between two vectors is small relative to their lengths makes it clear that it is equivalent to earlier presentations. One finds without much effort that \( \alpha^m \beta \) iff for a.e. \( x \)

\[
\limsup_{N \to \infty} \frac{1}{\# B_N} \sum_{\vec{v} \in B_N} \tau(h_{x,\beta}(\vec{v}), \vec{v}) = 0.
\]
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Although the vocabulary of [6] is somewhat different it is shown there that this is equivalent to saying:

**Proposition 2.3.5.** For every $\varepsilon > 0$ there is a $\phi \in \Gamma$ and a subset $A$ with $\mu(A) > 1 - \varepsilon$ so that for all $x, y \in A$,

$$\tau(\phi(x,y), \alpha(x,y)) < \varepsilon.$$  

For $G = \mathbb{Z}$ this implies $T^{\alpha \phi}$ and $T^\beta$ induce the same map on $A$ and hence $T^\alpha$ and $T^\beta$ are evenly Kakutani equivalent in the classical sense. In [6] the converse of this is proven, i.e. this is precisely even Kakutani equivalence in $\mathbb{Z}$ and a broad exploration of this equivalence relation in $\mathbb{Z}^d$ is made connecting it to Katok cross-sections of $\mathbb{R}^d$ actions.

As $m^K$ is entropy preserving we know the Bernoulli actions are $m^K$-finitely determined and hence there exist $m^K$-finitely determined actions. By Theorem 7.2.6 they are characterized by the condition of being weakly $m^K$-finitely determined. Notice that for actions of $\mathbb{Z}$ this precise fact is proven in Lemma 4.3 of [54].

**Example 3 (\(\alpha\) equivalences).**

Once more take $G = \mathbb{Z}^n$ and choose a vector $\bar{\alpha} = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of nonzero real numbers. Set $A : \mathbb{R}^n \to \mathbb{T}^n$ to be

$$A(v_1, \ldots, v_n) = (v_1/\alpha_1, v_2/\alpha_2, \ldots, v_n/\alpha_n) \mod \mathbb{Z}^n$$

and on $\mathbb{T}^n$ use the natural metric

$$\rho(\bar{v}_1, \bar{v}_2) = \|e^{2\pi i \bar{v}_1 \cdot \bar{v}} - e^{2\pi i \bar{v}_2 \cdot \bar{v}}\|$$

Notice that for $\rho \circ A$ to be small means the two vectors differ approximately by a vector $\{n_1\alpha_1, \ldots, n_i\alpha_i\}$ where the $n_i$ are integers.

For $q \in \mathcal{B}$ set $A(q) = \rho(A(q(\bar{0})), A(\bar{0})). A$ is not a size kernel as it fails to satisfy 1) although it does satisfy both 2) and 3). To obtain a size kernel all we need do is add to $A$ some other size kernel. We currently have two choices giving the two size kernels

$$D_\bar{\alpha}(q) = d(H(q), \text{id}) + A(q) \text{ and}$$

$$K_\bar{\alpha}(q) = K(q) + A(q).$$

(Notice it makes no sense to use $d(q, \text{id})$ as adding on $A(q)$ would add no further restriction to the already minimal equivalence class.)

Both of these examples give interesting equivalence relations. The second has been well-studied under the name of $\alpha$-equivalence (see [7] and [41]). Because of the standard use of $\bar{\alpha}$ to represent the parameter of this relation we will use $\beta$ to represent an arrangement throughout the discussion of these two examples.
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The first example, $D_T$, has not been discussed in the literature so we present a brief discussion here. What we obtain is a refinement of simple orbit equivalence that splits the ergodic actions into a countable list of equivalence classes characterized spectrally.

Remember a function $f : X \to \mathbb{C}$ is an eigenfunction of the ergodic action $T$ with eigenvalue $\lambda$ if $f$ is of norm one and

$$f \circ T = e^{2\pi i \sigma \lambda} f.$$  

For the vector $\lambda_0 = (1/\alpha_1, 1/\alpha_2, \ldots, 1/\alpha_n)$, those values $(k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n$ for which $(k_1/\alpha_1, k_2/\alpha_2, \ldots, k_n/\alpha_n)$ is an eigenvalue for $T$ form an additive subgroup we will call $\Lambda_\sigma(T)$. Two ergodic actions $U$ and $V$ of $\mathbb{Z}^n$ are $m^{D_A}$ equivalent iff $\Lambda_\sigma(U) = \Lambda_\sigma(V)$. In particular we see that there are at most countably many $m^{D_A}$ equivalence classes. We will indicate the proof of parts of this characterization leaving much to the reader.

**Proposition 2.3.6.** If $\beta_1 \overset{m^{D_A}}{\sim} \beta_2$ then $\Lambda_\sigma(T^{\beta_1}) = \Lambda_\sigma(T^{\beta_2})$.

**Proof.** Suppose $f$ is an eigenfunction of $T^{\beta_1}$ with eigenvalue $\lambda = (k_1/\alpha_1, k_2/\alpha_2, \ldots, k_n/\alpha_n)$. We now compute that for all $\tilde{v}$

$$\|f \circ T^{\beta_1}_{\tilde{v}} - e^{2\pi i \sigma \lambda} f\|_1 = \|f \circ T^{\beta_1}_{\tilde{v}} - f \circ T^{\beta_1}_{\tilde{v}}\|_1 = \|f \circ T^{\beta_1}_{\tilde{v}}(x, \phi(x)) - f \circ T^{\beta_1}_{\tilde{v}}(x, \phi(x))\|_1 \leq 2m^{D_A}(\beta_1, \phi).$$

Thus if $\beta_1 \overset{m^{D_A}}{\sim} \beta_2$, for all $\tilde{v}$ we will have

$$\|f \circ T^{\beta_2}_{\tilde{v}} e^{-2\pi i \sigma \lambda} - f\|_1 \leq 2m^{D_A}(\beta_1, \beta_2).$$

By the mean ergodic theorem there must be an $\hat{f}$ with

$$\frac{1}{\#B_N} \sum_{x \in B_N} f \circ T^{\beta_2}_{\tilde{v}} e^{-2\pi i \sigma \lambda} \to \hat{f}$$

and by the above,

$$\|f - \hat{f}\|_1 \leq 2m^{D_A}(\beta_1, \beta_2).$$
We know \( \hat{f} \) must have constant norm and as long as it is not identically 0, \( \hat{f}/|\hat{f}| \) will be an eigenfunction for \( T^{\beta_1} \) with eigenvalue \( \lambda \). As all \( T^{\beta_1} \) are conjugate to \( T^{\beta_1} \), we can assume \( m^{D_\alpha}(\beta_1, \beta_2) < 1/2 \) forcing \( \hat{f} \neq 0 \).

\[ \square \]

**Lemma 2.3.7.** Suppose \( T^\beta \) is a free and ergodic action of \( \mathbb{Z}^n \) on \( (X, \mathcal{F}, \mu) \) with \( \Lambda_\beta(T^\beta) = \{ 0 \} \), and \( P : X \to \Sigma \) is a finite partition. For each \( \varepsilon > 0 \) there is a \( \delta > 0 \) so that for any other free and ergodic action \( S^\gamma \) on \( (Y, \mathcal{G}, \nu) \) and partition \( Q : Y \to \Sigma \) satisfying:

1. \( \| (T^\beta, P), (S^\gamma, Q) \|_* < \delta \)
   and any value \( \delta_1 < 0 \) there is a \( \phi \) in the full group of \( S^\gamma \) and a partition \( Q' : Y \to \Sigma \) with
   a. \( m^{D_\alpha}(\gamma, \phi) < \varepsilon \)
   b. \( \nu(Q \Delta Q') < \varepsilon \) and

1'. \( \| (T^\beta, P), (S^\gamma \phi, Q') \|_* < \delta_1 \).

We once more leave a complete proof of this copying lemma to the reader. We do point out the ingredient used to obtain a. beyond the construction of Lemma 2.2.3. For \( \sigma \) fixed consider the group rotation on \( \mathbb{T}^n \) given by \( (x_1, \ldots, x_n) \to (x_1 + \alpha_1, \ldots, x_n + \alpha_n) \mod 1 \). This is not necessarily ergodic but all its ergodic components are conjugate to some group rotation we call \( (R_\delta, Z) \) where \( Z \) is a compact subgroup of \( \mathbb{T}^n \). To say \( \Lambda_\beta(T^\beta) = \{ 0 \} \) is equivalent to saying \( R_\delta \times T^\beta \) is ergodic. Partition \( Z \) into sets of diameter less than \( \varepsilon/2 \) by a partition \( H \).

Consider now \( H \vee P, B_N \)-names arising from the action of \( R_\delta \times T^\beta \). The pointwise ergodic theorem tells us that if we fix a choice of \( h \in H \) and cylinder set \( C \) in the process \( (T^\beta, P) \) then for \( N \) large the relative density of \( C \) just at indices of an \( H \vee P, B_N \)-name whose \( H \) term is \( h \) will be very close to \( \mu(C) \).

Fixing the value \( N \), if \( (S^\gamma, Q) \) satisfies 1. for a small enough \( \delta \) then this same fact will be true for \( H \vee Q \)-names relative to \( \nu \) (even if \( R_\delta \times S^\gamma \) is not ergodic). The full group element \( \phi \) will now be constructed on some Rohklin tower of size \( B_M, (M >> N) \) in the action \( S^\gamma \) by overlaying the \( S^\gamma, Q \) names with some template \( R_\delta, H \)-name and constructing \( \phi \) to move this name close in \( d \) to some \( T^\beta, P \)-name while simultaneously preserving the template name. The remark in the previous paragraph guarantees that this can be done. This preservation of the template name is the new ingredient needed to
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ensure $m^{D_\alpha}(\gamma, \phi) < \varepsilon$. One now changes $Q$ slightly according to the necessary $\tilde{d}$ error to be an exact copy of the target $T^\beta, P$-name.

This lemma and the previous Proposition now imply that the $m^{D_\alpha}$-finitely determined actions are those with $\Lambda_{\tilde{d}}(T^\beta) = \{0\}$ (see Definition 7.2.5 and Theorem 7.2.6) and any two such actions will be $m^{D_\alpha}$-equivalent.

One approach to showing that the group $\Lambda_{\tilde{d}}(T^\beta)$ is a complete invariant even outside the finitely determined class is to fix a choice $\Lambda$ for $\Lambda_{\tilde{d}}$ and generalize Lemma 2.3.7 and the theory of $m^{D_\alpha}$-joinings of Chapter 6 to a relativized version relative to the nontrivial isometric factor algebra generated by the eigenfunctions whose eigenvalues lie in $\Lambda$.

As indicated earlier the size kernel $K_{\tilde{d}}$ leads to a rather well-studied area. Certainly $m^{K_{\alpha}}$-equivalence refines even Kakutani equivalence. The argument in Proposition 2.3.6 applies here as well to say the group $\Lambda_{\tilde{d}}$ is an $m^{K_{\alpha}}$ invariant. Hence each even Kakutani equivalence class is cut into at least countably many $m^{K_{\alpha}}$ classes. In [7] it is shown that in one dimension, and for the “loosely Bernoulli” class (the $m^{K_{\alpha}}$-finitely determined class) this is the full refinement. That argument will push through to all dimensions. As a consequence of the construction in [13] there are other $m^K$ classes which contain uncountably infinitely many $m^{K_{\alpha}}$ classes.

There exists a connection to sections for actions of $\mathbb{R}^n$ as well, although it is understood only in 1 and 2 dimension. Here is what is known in one dimension. For $\alpha$ irrational and $\beta > 0$ any measure preserving flow can be represented as a flow under a function taking on only the two values 1 and $1 + \alpha$ [38]. Just as for Kakutani equivalence itself one can define here a relationship between $\mathbb{Z}$ actions by saying they are $\alpha$-related if they arise as the return maps from a common flow where the return times take on only these two values 1 and $1 + \alpha$. We say they are evenly $\alpha$-related if the integrals of these return time functions agree. In [7] it is shown that two $\mathbb{Z}$ actions are $\alpha$-related iff they are $m^{K_{\alpha}}$-equivalent. Moreover it is shown that if $U$ and $V$ are $m^{K_{\alpha}}$ equivalent, then any flow for which $U$ arises as such a section, $V$ does as well.

Here is what is known in two and higher dimensions. In [40] it is shown that any $\mathbb{R}^n$ action can be represented as a special sort of “Markov” tiling suspension of an action of $\mathbb{Z}^n$ where the tiles are rectangles whose length in dimension $k$ is either only 1 or $1 + \alpha_k$ (we assume all $\alpha_k$ are irrational and $\beta > 0$). We say two actions $U$ and $V$ of $\mathbb{Z}^n$ are $\tilde{\alpha}$ related if they arise as such sections of a common $\mathbb{R}^n$ action. We say they are evenly $\tilde{\alpha}$-equivalent if the proportion of the space occupied
by each of the tile shapes is the same for both representations. The argument in [7] extends to higher dimensions to show that if $U$ and $V$ are evenly $\tilde{\alpha}$ related then they are $m^K\tilde{\alpha}$ equivalent. In two dimensions Sahin [41] shows that the converse is true, i.e. if two actions $U$ and $V$ are $m^K\tilde{\alpha}$ equivalent then they arise such Markov tiling sections of a common $\mathbb{R}^n$ action. It remains open however whether any $\mathbb{R}^n$ action for which $U$ is such a section must also have $V$ as such a section.

Our last two examples exhibit another general context in which a restricted orbit equivalence relation can arise. Suppose to each arrangement $\alpha$ we can assign a subgroup $\Gamma_0^\alpha$ of the full group $\Gamma$ with the equivariance property that $\Gamma_0^{\alpha\phi} = \phi^{-1}\Gamma_0^\alpha \phi$. In particular the choice of subgroup does not change when we perturb $\alpha$ by an element of its subgroup. What interests us are those $\beta$ reachable as limits of sequences of rearrangements $\alpha\phi_i$ where $\phi_i \in \Gamma_0^\alpha$. We write a size for such a relation as a sum of two pieces, one measuring the $m_0^\alpha$ distance from $\phi$ to $\Gamma_0^\alpha$ and the other measuring some chosen size of $(\alpha, \phi)$. As a simple example of this consider $\alpha$ equivalence in $\mathbb{Z}$ for $\alpha = 2$. Here the groups $\Gamma_0^\alpha = \{\phi|\phi(x, \phi(x)) = 0 \mod 2\}$. Using the two sizes $m^K$ and $m^K$ within the subgroups will yield the two examples described of $\tilde{\alpha}$-equivalence. This idea becomes cumbersome for irrational $\alpha_i$. We do not attempt to axiomatize precisely what is needed of the family of subgroups, leaving this to a more general study of sizes. Our final two examples will offer an indication of the range this idea covers.

**Example 4** (Vershik’s $r$ Equivalence).

The work described here can be found in [16] and [15] of D. Heicklen. Suppose $(X, \mathcal{F}, \mu)$ is a standard nonatomic probability space and $\mathcal{F}_i$ is a sequence of sub $\sigma$-algebras with $\mathcal{F} = \mathcal{F}_0$ and $\mathcal{F}_{i+1} \subseteq \mathcal{F}_i$. We refer to such a sequence as a **reverse filtration**. Two such are **conjugate** if there is a measure preserving bijection between the measure spaces carrying one reverse filtration, term by term, to the other. To remove some trivial issues and make this subject addressable by our methods we make two assumptions. First, we assume that for each $i$ the conditional fiber measures of $\mathcal{F}_i$ over $\mathcal{F}_{i-1}$ are atomic with a fixed number of atoms $k_i$ and that each atom has a constant mass $1/k_i$. We call such a filtration **uniform**. Next we assume that the $\mathcal{F}_i$ intersect to the trivial algebra. A filtration with this property is called **exact**. One natural way for such a filtration to arise is from an action of a group of the form $G = \sum_{n=1}^\infty \mathbb{Z}/r_n\mathbb{Z}$. What matters here is that $G$ is the increasing union of the finite groups $H_i = \sum_{n=1}^i \mathbb{Z}/r_n\mathbb{Z}$. If we have a measure preserving and free action of this group and we set $\mathcal{F}_i$ to
be the algebra of $H_i$ invariant sets then we obtain a uniform reverse filtration. It is exact iff the action is ergodic. Conversely given any uniform and exact reverse filtration, using the Rokhlin decomposition of each successive $\mathcal{F}_i$ over $\mathcal{F}_{i-1}$ we can place on the space an action of $G$ for which the filtration is obtained as this list of invariant sub-algebras. The action of $G$ here is not unique and this leads to a natural relation: We say two actions of $G$ are Vershik related if conjugate versions of both of them can be placed on the same space, giving rise to the same reverse filtration. Notice in particular that the two actions will be orbit equivalent and what characterizes the particular orbit equivalence is that it preserves the orbits of all the subgroups $H_i$.

For the purpose of our discussion it will be useful to assume only that $G$ is the increasing union of finite abelian groups $H_i$ without assuming that each is cyclic over its predecessor. Such a $G$ is countable and amenable. Notice that for any subsequence of indices $1 \leq j_1 < j_2 < \ldots$ we could define $\hat{H}_i = H_{j_i}$ and get another representation of $G$ as an increasing union of finite subgroups. These distinct representations will give distinct values for the vector $\vec{r} = \{ |\hat{H}_1|, |\hat{H}_2/\hat{H}_1|, \ldots \}$ and so we can represent a choice for such an increasing subsequence of subgroups by its vector $\vec{r}$. This is consistent with the usage when $H_i/ H_{i-1}$ is cyclic of order $r_i$. We say two actions of $G$ are Vershik $\vec{r}$-related if they are Vershik related for the choice of subgroups $\hat{H}_i$ determined by the values of $\vec{r}$. We describe Vershik relatedness indexed by the choice of subsequence $\vec{r}$ as a family of restricted orbit equivalences on $G$.

Although one can use a size kernel here we follow [16] and give the size directly. Notice that for a fixed arrangement $\alpha$ and choice for $\vec{r}$ the full group $\Gamma$ contains closed subgroups $\Gamma_{r_{0}^{\alpha}}\phi$ consisting of those $\phi$ which preserve the $T^{\alpha}$ orbits of all $\hat{H}_i$. (This is equivalent to saying that either $q_{\alpha}^{\phi}$ or equivalently $h_{\alpha}^{\phi}$ permutes cosets of $\hat{H}_i$ for all $i$ and a.e. $x$.) If we have a sequence of rearrangements $\alpha_{\phi_i}$ converging to some $\beta$ where all the $\phi_i \in \Gamma_{r_{0}^{\alpha}}$ then $T^{\alpha}$ and $T^{\beta}$ will have identical $\hat{H}_i$ orbits for all $i$ and hence be Vershik $\vec{r}$ related in this very strong sense. To define a size giving this relation, for $\alpha$ and $\vec{r}$ fixed we first calculate the distance some $\phi$ is from the subgroup $\Gamma_{r_{0}^{\alpha}}\phi$ as its $m_{\alpha}$ distance:

$$c_{\phi}(\alpha,\phi) = \inf_{\phi' \in \Gamma_{r_{0}^{\alpha}}} \mu_{\{x | \phi(x) \neq \phi'(x)\}}.$$

(This is not the definition of $c_\phi$ given in [16] but is equivalent by the Flattening Lemma proven there.) One can now define the family of sizes
\[ m^\vphi(a, \phi) = c^\vphi(a, \phi) + m^0(a, \phi). \]

Heicklen proves this to be a size but does not show it to be \( 3^+ \). We will not present the details that it is in fact \( 3^+ \). This can be done either by suitably expanding Heicklen’s argument or by showing the sizes \( m^\vphi \) arise from size kernels.

Heicklen’s conclusion is that two actions \( \alpha \sim^m \beta \) iff there is a \( \psi \in \Gamma \) so that \( T^{\alpha \psi} \) and \( T^\beta \) have identical \( \hat{H}_i \) orbits for all \( i \). As \( T^\alpha \) and \( T^{\alpha \psi} \) are conjugate (by \( \psi \) of course) \( T^\alpha \) and \( T^\beta \) are \( \hat{r} \) related and if two actions are \( \hat{r} \) related they can be realized as two such actions.

The family of sizes exhibits two very interesting properties. The first is due to Vershik who proved a **lacunary isomorphism theorem** for such groups [50]: For any two actions \( U \) and \( V \), if the \( r_i \) are chosen to grow rapidly enough, then the two actions are \( \hat{r} \) related. Notice that the allowed choices for the sequence \( \hat{H}_i \) are partially ordered under containment. As one goes further out in this partial order more and more actions become equivalent and Vershik’s result says any two will become equivalent once one is far enough out in this net.

The second very interesting property follows from this. Vershik has shown that for very slowly growing sequences, like \( r_i = 2 \) for all \( i \), the entropy of a \( G \) action is an invariant of \( \hat{r} \) equivalence. On the other hand, Vershik’s lacunary isomorphism theorem tells us that for some choices of \( \hat{r} \), entropy definitely is not an invariant. As we learn in Chapter 5 a restricted orbit equivalence either preserves entropy or generically in the \( m_\alpha \) topology an action has zero entropy. Vershik [51] conjectured and proved the sufficiency and Heicklen [16] proved the necessity of the following characterization of the boundary between these two regimes: The size \( m^\vphi \) is entropy preserving iff

\[ \sum_{i=1}^{\infty} \frac{\log r_{k+1}}{\# H_k} < \infty. \]

**Example 5 (Entropy as a Size).**

We discuss this example only for actions of \( \mathbb{Z} \) although the ideas extend to general countable amenable groups. The results described here are found in [39]. Before examining this example in detail consider the following observations. Two major goals of this current work are to demonstrate:

1. A size \( m \) is either entropy preserving in that two equivalent actions have the same entropy, or entropy free in that residually in each class actions have zero entropy. In the first case we say
an action’s $m$-entropy is its entropy and in the latter that its $m$-entropy is always zero.

2. Each size possesses a family of distinguished classes, characterized by their $m$-entropy, called the $m$-finately determined classes. Any two $m$-finately determined actions of the same $m$-entropy are $m$-equivalent.

Notice that this implies the possibility of two sizes $m$ for which all actions are finitely determined, one that is entropy free and one that is entropy preserving. Dye’s theorem, here done via the size $m^0$ shows that there is an entropy free size for which all actions are $m$ finitely determined. What the example we now discuss shows is that the other size also exists, relative to which two actions are equivalent iff they have the same entropy.

The size at its base will simply be the entropy of the rearrangement itself. We make this precise as follows. The function $g(\alpha, \phi(x)) = \alpha(x, \phi(x))$ takes on countably many values and hence can be regarded as a countable partition $g(\alpha, \phi)$ of $X$. Set $\Gamma_0^\phi$ to be those $\phi$ for which $g(\alpha, \phi)$ is finite. It is not difficult to see that $\Gamma_0^\phi$ is a subgroup and moreover $\Gamma_0^\phi = \psi^{-1}\Gamma_0^\psi$ as $g(\alpha, \psi(\phi^{-1}(x))) = g(\alpha, \phi(x))$. It is shown in Theorem 4.0.27 that the $\Gamma_0^\phi$ are all $m_\alpha^1$ dense in $\Gamma$. For $\phi \in \Gamma_0^\phi$ one can use the entropy of the process $h(T^\alpha, g(\alpha, \phi))$ to start the definition of a size defining

$$e(\alpha, \phi) = \inf_{\phi' \in \Gamma_0^\phi} h(T^\alpha, g(\alpha, \phi')) + \mu\{x|\phi(x) \neq \phi'(x)\}.$$

Now set the size to be

$$m^e(\alpha, \phi) = e(\alpha, \phi) + m^0(\alpha, \phi).$$

To see that this is a size, Axiom 1 follows from basic conditional entropy considerations and Axiom 2 is directly due to the second term. Axiom 3 here follows from upper semi continuity of entropy and for this reason this is not on the face of it a $3^+$ size. To see that in fact it is not, note that those $\alpha$ for which $T^\alpha$ is zero entropy are $m_0^0$ residual. Hence a rearrangement $(\alpha, \phi)$ can be perturbed by as little as we like in distribution to an $(\alpha', \phi')$ with $m^e(\alpha', \phi') = 0$. It is this example which motivated the weakening of Axiom 3 to Axiom 3.

The form of the size makes it reasonable to believe and easy to prove that $m^e$-equivalence will be entropy preserving. A more subtle combinatorial argument leads to the reverse conclusion as well, that any two ergodic actions of equal entropy are in fact $m^e$-equivalent.