M417, Fall 2009, First hourly exam

This is a closed book, closed notes exam. Show all your work. For full credit you must show complete arguments.

Prob. 1 (30 pts)

Suppose \( f : \mathbb{R}^2 \to \mathbb{R} \) and we find that the directional derivative in the direction \( \vec{u} = (u_1, u_2) \) at the point \((1, 2)\) is given by
\[
\partial_{\vec{u}} f(1, 2) = u_1 + u_2^2.
\]

a) What is \( \nabla f(1, 2) \)?

b) Is \( f \) differentiable at \((1, 2)\)? Why or why not?

Prob. 2 (35 pts)

A sequence is defined recursively by \( x_1 = 1 \) and \( x_{k+1} = \sqrt{1 + x_k} \).

a) Show that the \( x_k \) are monotone increasing and bounded above by 3 and hence converge.

b) Find the limit of the \( x_k \).

c) If \( x_1 \) is changed to 3 does the sequence still converge? Why or why not?

Prob. 3 (35 pts)

Suppose \( f : (0, 1] \to \mathbb{R} \) is continuous and the \( \lim_{x \to 0^+} f(x) = L \) exists. Prove that \( f \) is uniformly continuous.