Sept 25

Differentiation

Approximation

\( f : \mathbb{N} \to \mathbb{R} \)

\( \mathbb{N} \subseteq \mathbb{R} \)

\( a \in \mathbb{N}^{*} \)

Approx. \( f \) near \( a \).

by polynomials

0'th deg poly.

\[ c \cdot x = c \]

What is the best choice for \( c \) to approx. \( f \) near \( a \)?
We'll try f(a).

Is this a good choice?

\[
\lim_{x \to a} \left| \frac{f(x) - c(x)}{x - a} \right| = \lim_{x \to a} \frac{|f(x) - f(a)|}{|x - a|}
\]

If f is cont. then this = 0 + is a reasonable approx.

Let order approx.

\[L(x) = c + m(x-a)\]

\[c = f(a)\]

\[= f(c_a) + m(x-a)\]

\[
\lim_{x \to a} \left| \frac{f(x) - L(x)}{x - a} \right| = 0
\]

in order for the approx to be good.
\[ \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = m. \]

Just saying

\[ \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = m. \]

\[ f'(a) \]

1) If \( a \in D \) and \( f \) has a max or min at \( a \) and \( f \) is differentiable at \( a \), then \( f'(a) = 0 \)

2) Rolle's Thm. If \( f \) is continuous on \([a, b]\), differentiable on \((a, b)\)

\[ f(a) = f(b) \]
Theorem (Generalized Mean Value Theorem)

Suppose \( f, g \) are continuous on \([a, b]\) and differentiable on \((a, b)\).

\[ f'(c) = \frac{f(b) - f(a)}{g(b) - g(a)} \]

Then, \( \exists \ c \in (a, b) \)

\[ h(x) = (g(x) - g(a))f(b) - f(a) - (f(x) - f(a))(g(x) - g(a)) \]

\[ h(a) = 0 \]
\[ h'(c) = 0 \]

\( h \) is constant on \([a, b]\) and differentiable on \((a, b)\)

\( \exists \ c \in (a, b), \ h'(c) = 0 \)
\[ h'(x) = g'(x)(f(b) - f(a)) \]
\[ - f'(x)(g(b) - g(a)) \]

\[ \frac{g'(c)(f(b) - f(a))}{g'(c)(g(b) - g(a))} = \frac{f'(c)}{g'(c)} \]

\[ g(b) \neq g(a) \text{ by} \]

Rolle's Theorem
\[ (\text{if } = \Rightarrow \exists \delta \in (a, b) f'(x) = 0) \]

\[ \frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)} \]

\[ f: \mathbb{R} \rightarrow \mathbb{R} \]
\[ f = (f_1, f_2, \ldots, f_n) \]

To say \( f \) is differentiable at \( a \) should mean
\( f \) has a good linear approximation.
\[ L(x) = (f_1(a) + m_1(x-a), f_2(a) + m_2(x-a), \ldots, f_n(a) + m_n(x-a)) \]

\[
\lim_{x \to a} \frac{\| f(x) - L(x) \|}{|x-a|} = 0
\]

\(\iff\) \(\lambda = 1, \ldots, n\)

\[
\lim_{x \to a} \frac{\| f(x) - f_\lambda(a) \|}{x-a} = 0.
\]

\(\iff\) each \(f_\lambda\) is diff.

Such \(f_\lambda\)'s are \(n\)-arps
\(\text{curves in } \mathbb{R}^n\)
\(\text{derivative is a vector } (f_1', f_2', \ldots, f_n')\)

\[ L(x) = f(a) + f'(a)(x-a) \]
\[ f : \mathbb{R}^n \to \mathbb{R} \]

**1st order polynomials in \( m \) variables**

\[ L (x) = f (\bar{a}) + c_1 (x_1 - \bar{a}_1) + c_2 (x_2 - \bar{a}_2) + \cdots + c_m (x_m - \bar{a}_m) \]

\[ = f (\bar{a}) + \overrightarrow{c} \cdot (x - \bar{a}) \]

**Linear approximation point**

**I want**

\[ \lim_{\| x - \bar{a} \| \to 0} \left( \frac{f (x) - L (x)}{\| x - \bar{a} \|} \right) = 0 \]

**Computing \( \epsilon_1 \)**

For \( \epsilon_1 \),

Look along the \( \epsilon_1 \) direction.

\[ \lim_{h \to 0} \left( \frac{f (\bar{a} + h \epsilon_1) - f (\bar{a})}{h} \right) - \epsilon_1 \]
\[
\begin{align*}
\frac{f(\mathbf{a} + h \mathbf{e}_i) - f(\mathbf{a})}{h} &= f(a_1 + h, a_2, a_3, \ldots, a_n) - f(a_1, a_2, a_3, \ldots, a_n) \\
&= \frac{f(a_1 + h, a_2, a_3, \ldots, a_n) - f(a_1, a_2, a_3, \ldots, a_n)}{h} \\
&= \text{So this limit being 0 means just that}\end{align*}
\]

\[
\left. c_i \right|_{a_i} = \frac{\partial f}{\partial x_i}(a_i)
\]

\[
\left. c_i \right|_{\mathbf{a}} = \frac{\partial f}{\partial x_i}(a_i)
\]

\[
L(x) = f(\mathbf{a}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a_i)(x_i - a_i) \]

\[
\sum_{i=1}^{n} \left( \frac{\partial f}{\partial x_i}(a_i)(x_i - a_i) \right) + \frac{\partial f}{\partial x_i}(a_i)(x_i - a_i)
\]