Compact Sets

$C \subset \mathbb{R}^n$

an "open cover" for $C$ is a collection $U$ of open sets with $C \subseteq \bigcup U$.

A subcover $V$ is a subset of $U$ that still covers $C$.

A "finite subcover" is one that has only finitely many elements.
Def A set \( C \) is called compact if every open cover of \( C \) has a finite subcover.

**Lemma**

a) If \( C \) is compact then \( C \) is bounded.

b) If \( C \) is compact then \( C \) is closed.

**Proof**

\( C \) is unbounded.

\[ U_n = B(0, \infty) \]
\[ U_n U_n = \mathbb{R}^m \quad \forall n \]

**Cover.** If

\[ \bigcup_{n=1}^{N_1} U_1 \cap U_2 \cap \ldots \cap U_N = C \]

cover \( C \).

\[ R = \max(N_1, \ldots, N_N) \]

\[ C \subseteq B_{R}(\bar{a}) \]

\[ \Rightarrow C \text{- bounded} \]

b) \( C \) is *not* closed.

so \( \exists \bar{a} \in \partial C \), \( \bar{a} \not\in C \).

\[ U_n = \left(\left( B_{\frac{1}{n}}(\bar{a}) \right) \right) \quad \forall n \]

so

\[ U = \bigcup_{n=1}^{\infty} U_n = \mathbb{R}^m - \sum a_i z_i \]

A cover - \( \bigcup_{\mathcal{J}} \) \( F \)
$U_{\eta_1} \cdots U_{\eta_k}$ cover $C$, then

Let $r = \min \{ \frac{1}{\eta_1}, \ldots, \frac{1}{\eta_k}\}$

$B_r(\overline{a}) \cap C = \emptyset$

$\Rightarrow \overline{a} \notin \partial C$.

$\Rightarrow \subseteq$

**Thm.** If $C \subseteq \mathbb{R}^n$ is closed & bounded

then it is cprf.

(Heine-Borel Thm.)

**Def.** We say $C$ is "sequentially cprf"
if any sequence 

\[ a_i \] with values in \( C \)

has a convergent subsequence 

\[ a_{k_i} \to \bar{a} \in C. \]

Then \( C \subseteq \mathbb{R}^n \), i.e.,

sequence \( c_{p_{k_i}} \) if it

is closed and bounded.

Proof

a) \( C \) - unbounded

\[ \Rightarrow \] not sequence \( c_{p_{k_i}} \).

Choose \( a_{i} \), \( \| a_{i} \| > i \).

b) \( C \) - not closed

\[ \Rightarrow \] \( C \) - not sequence \( c_{p_{k_i}} \).

\[ \exists \bar{a} \in \partial(C), \bar{a} \notin C. \]

\[ \bar{a} \in \partial(C) \Rightarrow \exists a_{i}, \bar{a} \in C, \]

\[ a_{i} \to \bar{a}. \]
as \( \bar{a} \in C \), no subs. of the \( \bar{a} \)'s can conv. to a pt of \( C \) - they all conv. to \( \bar{a} \).

c) \( C \) - closed + b.ded \( k \)

\( a_i \in C \); \( a_i \) has a conv. subregu. as \( a_i \) is b.ded

\( \bar{a}_i \rightarrow \bar{a}, \bar{a} \)

if \( \bar{a} \in C \) - done.

if \( \bar{a} \notin C \) -

\( \bar{a} \in dC \subseteq C. \)

\( \Rightarrow \subseteq. \)

Thm: Suppose

\( C \subseteq \mathbb{R}^n \), is \( \rho \)-c.

\( f: C \rightarrow \mathbb{R}^m \)
is continuous.

Then \( f(C) \) is CPCF.

\[ f(C) \]

Suppose \( \mathcal{V} \) is an open cover for \( f(C) \).

Let
\[ \mathcal{V} = \bigcup_{u \in \mathcal{V}} f^{-1}(u) \]

\( \mathcal{V} \) is an open cover for \( C \).

CPCF So \( \exists \)
\[ V_1, \ldots, V_k \] cover \( C \).

\[ \Rightarrow \]
\[ f^{-1}(u_i) \] \( \Rightarrow \)
\[ f^{-1}(u_i) = f(f^{-1}(A)) = A \]
\[ \Rightarrow U_1, \ldots, U_k \] cover \( C \).
2nd pf

Show \( f(c) \) is seq. c.p.c.t.

Let \( a_i \in f(c) \)

Let \( b \in C \), \( f(b) = c \).

\( b \) has a conv subs.

\( b \rightarrow \overline{b} \in C \).

\( f \) is conv \( a + \overline{b} \).

\( \overline{c} = f(\overline{b}) \rightarrow f(\overline{b}) \in f(c) \).
If $C \subseteq \mathbb{R}^n$ is compact and $f : C \rightarrow \mathbb{R}$ is continuous, then $f$ achieves a maximum and minimum on $C$.

If $f(C)$ is compact, hence closed and bounded, so $f(C)$ has a maximum and minimum.