Subsequences

\[ a_k \text{ - sequence} \]

\[ \bullet \quad \bullet \quad \bigcirc \quad \bigcirc \quad \bullet \quad \bullet \quad \bigcirc \]

Let \( k : \mathbb{N} \to \mathbb{N} \)
\[ k^1 > k^2 \]

then
\[ a_{k^i} = a_{k^i} \text{ is a } \]
\[ \text{subsequence} \]

Lemma: If
\[ \lim_{n \to \infty} a_n = L \]
\[ a_k \text{ is a subsequence} \]

then \( \lim_{i \to \infty} a_{k^i} = L \).
Def: We say \( a_k \) is "increasing" if \( a_{k+1} > a_k \).

\( a_k \) is "decreasing" if \( a_{k+1} < a_k \).

Monotone if it is one of these.

Lemma

1) If \( a_k \) is increasing and bounded above then it converges.
2) If \( a_k \) is decreasing and bounded below then it converges.

pf \[ L = \sup \{ a_k \} \]

Let \( \varepsilon > 0 \), \( \exists k \),

\[ a_k \leq L < a_k + \varepsilon. \]
\[ k \geq k \]

\[ a_k \leq a_k \leq \frac{1}{k} < a_k + \epsilon \leq a_k + \epsilon \]

\[ \Rightarrow |L - a_k| \leq \epsilon. \]

\[ a_k \rightarrow L \]

**Theorem.** Every \( \mathbb{R} \)-valued sequence has a monotone subsequence.

**Proof.**

**Case 1.** \( a_k \) is unbounded above. 
\[ \exists \text{ subseq } \to \infty. \]

**Case 2.** \( a_k \) is unbounded below. 
\[ \exists \text{ subseq } \downarrow -\infty \]

**Case 3.** \( a_k \) are bounded. 
\[ \text{sup}(\{a_n\}) \text{ is not a max.} \]
Case 4 \( a \) is bounded \( \chi \) has a subsequence \( a_{n_k} \) whose sup is \( \mathrm{max} \).

\[ \sup \left( \epsilon_0, \epsilon \right) \]

Case 5 \( a \) is bounded and for all subsequences, the sup is \( \mathrm{max} \).

\[ \sup \left( a_{k_1}, a_{k_2}, \ldots \right) = a_{k_2} \]
\[ \sup (a_k, a_{k+1}, a_{k+2}, \ldots) = a_{k+3} \]

\[ \ldots \]

\[ a_k \text{ decrease} \]

**Cor.** Every bounded sequence has a convergent subsequence.

Subsequential limit value

\[ a_k \xrightarrow{\text{subsequential limit value}} \]

\[ a_k \xleftarrow{\text{is a subsequential limit value}} \]

\[ a_k \xrightarrow{\text{is a subsequential limit value}} L \]

**Ex.**

\[ a_k = \sin \left( \frac{\pi k}{3} + \frac{1}{k} \right) \]
\[ k - k = 1 \mod 3 \]
\[ = 2 \mod 3 \]
\[ = 0 \mod 3 \]

\[ k = 1 \mod 3 \]
\[ a_k = \sin \left( \frac{2\pi}{3} + \frac{1}{k} \right) \rightarrow \frac{\sqrt{3}}{2} \]

\[ k = 2 \mod 3 \]
\[ a_k = \sin \left( \frac{4\pi}{3} + \frac{1}{k} \right) \rightarrow -\frac{\sqrt{3}}{2} \]

\[ k = 0 \mod 3 \]
\[ a_k = \sin \left( \frac{\pi}{3} \right) \rightarrow 0 \]

\[ \text{Ex} \]
\[ \frac{1}{2}, \frac{1}{2}, 1, 0, \frac{1}{2}, 0, \frac{1}{2}, 1, \frac{1}{2}, 0, \frac{1}{2}, 1, \frac{1}{2}, 0 \]

\[ a_k = \ldots \]
\[ B_m = \sum_{k=1}^{\frac{n}{m}} \frac{1}{m} \left( \frac{z}{m} \right) \]

\[ \alpha \in [0,1] \]

\[ \exists \left( \alpha_k \right)_m \subset (0,1) \]

\[ \left( \alpha_k - \alpha \right) < \frac{1}{m} \]

\[ k_m \rightarrow k_{m+1} \]

\[ \alpha_k \text{ is a subnet of } \alpha \]

\[ \left| a_k - \alpha \right| < \frac{1}{m} \]

\[ \lim_{m \rightarrow \infty} \left| a_k - \alpha \right| = 0 \]

\[ a_k \rightarrow \alpha. \]

**Hint**

\[ \lim_{m \rightarrow \infty} \frac{5}{a} \]

\[ [a, b] \]

\[ [a, a + \frac{b-a}{2}] \cup [a + \frac{b-a}{2}, b] \]

\[ I \]

\[ I_n \]
\[ I_{i_1} S_{i_1} = I_{i_2} S_{i_2} = I_{i_3} S_{i_3} = I_{i_4} S_{i_4} \]

Assume \( I_{i_1} S_{i_1} \) is new.

\[ I_{i_1} \cap I_{i_2} \cap I_{i_3} \cap I_{i_4} \]

\[ I_{i_1} \cup S_{i_1} = I_{i_2} \cup S_{i_2} \cup S_{i_3} \cup S_{i_4} \]

\[ x \in I_{i_1} \cap I_{i_2} \cap I_{i_3} \cap I_{i_4} \]