HW due: 18th
p. 23 #2, 6
p. 29 #5, 8

#3 \[ \sin x \frac{y}{x}, \quad x \neq 0 \]

\[
\begin{align*}
\mathbb{R}^+ & \to (0, y) \\
\text{sequences} & \quad (x_i, y_i) \to (0, y) \\
x_i & \neq 0, \\
F(x_i, y_i) & = \frac{\sin (x_i, y_i)}{x_i} = a_i \\
\lim_{i \to \infty} a_i & ?
\end{align*}
\]
\[
\frac{\sin(x_i y_i) - \sin(0)}{x_i y_i - 0} \quad \text{Thm} \\
\int_{a}^{b} f(x) \, dx = f(c) \\
\text{c is between a + h},
\]
\[
\frac{\sin(x_i y_i) - \sin(0)}{(x_i y_i) - 0} \\
\text{deriv of } \sin \text{ at some } c_i \text{ between } x_i y_i + 0.
\]
\[
= \cos(c_i) y_i
\]
\[ \lim_{i \to \infty} \cos(c_i) = 1 \]

\[ y_i \to y \]
\[ x_i \to 0 \]
\[ \tau_i y_i \to 0 \]
\[ \tau_i \theta_i \to 0 \]
\[ c_i \text{ is between } x_i y_i + 0. \]
\[ x_i y_i \to 0. \]
\[ \therefore \lim_{i \to \infty} c_i = 0 \]

\[ \lim_{i \to \infty} \cos(c_i) = 1 \]

\[ \lim_{i \to \infty} \theta_i = \theta \]

\[ \text{Three C's} \]

Completeness \< Compactness \< Connectedness
$\mathbb{R}$

3. Kinds of properties

1) arithmetic
2) order
3) completeness -

Classical picture -

$S$ - set $\subseteq \mathbb{R}$

It might have upper bounds.

$\mathcal{U}(S) = \{ b \in \mathbb{R} : \forall s \in S, \ b \geq s \}$

If $\mathcal{U}(S)$ is not $\emptyset$ we say $S$ is bounded above.

If $b \in \mathcal{U}(S)$

$b' \geq b$ then $b' \in \mathcal{U}(S)$
A “least upper bound” is $b_0 \in \mathbb{U}(S)$ such that for all $b \in \mathbb{U}(S)$,

$$b \geq b_0.$$  

Completeness prop. of $\mathbb{R}$ says any non-empty bounded subset $S \subseteq \mathbb{R}$ has a least upper bound.

If $S$ is not bounded then we set its l.u.b. $= \infty$.

If $S = \emptyset$, l.u.b. $= -\infty$

$$\sup(S) = \text{l.u.b.}(S)$$
\[ \text{inf}(s) = -\text{p.u.b.}(s) \]

If \( \sup(s) \leq S \)
then call it \( \text{max}(s) \).

If \( \inf(s) \leq s \)
then call it \( \text{min}(s) \).

Alternative:

\underline{Cauchy Sequences.}

Def. A sequence \( \{a_n\} \) is called Cauchy
if \( \forall \varepsilon > 0 \exists N, \forall n, m \geq N \)
all \( n, m \geq N \)
\[ \left| \frac{a_n}{a_m} - \frac{a_m}{a_n} \right| < \varepsilon \]

Lemma. If \( a_n \) converges
it is Cauchy.
Lemma: If \( a_n \) is Cauchy then it converges.

Proof: Assume \( a_n \in \mathbb{R} \).

a) As \( a_n \) are Cauchy, they are bounded.

Let \( \varepsilon = 1 \), \( \exists N \), \( m, n \geq N \), \( |a_m - a_n| < 1 \).

\[
\delta = \max \{ a_1, a_2, \ldots, a_N, a_{N+1} \}
\]

\[
m \leq N \quad \Rightarrow \quad a_m \leq \delta
\]

\[
m \geq N \quad \Rightarrow \quad |a_m - a_n| < 1
\]

so \( a_m < a_{m+1} \)

Let \( b = \sup \{ a_m, a_{m+1}, a_{m+2}, \ldots \} \)

\[
b_{n+1} \leq b_n
\]
Values \( b_n \) are bounded below.

Let \( L = \inf (b_n) \)

\[ \forall \varepsilon > 0 \quad \exists N, \ n, m \geq N \quad |a_n - a_m| < \varepsilon. \]

\[ b_n = \sup (\{a_n, a_{n+1}, \ldots \}) \]

so \( \forall n \geq N \)

\[ |b_n - a_n| < \varepsilon \]

The \( b_n \)'s decrease

\[ \inf b_n = L. \]

Since \( \inf b_n = L \),

\[ \forall \varepsilon > 0, \ \exists N, \ 0 \leq b_n - L \leq \varepsilon \]

\[ \text{if} \quad |b_n - L| \leq \varepsilon \]
\[ n > m, \quad b_n - \varepsilon \leq b_m \leq b_n + \varepsilon \]

Hence,

\[ \varepsilon \]

∀ \( N \) large enough

\[ 0 < b_N \leq \varepsilon \]

Moreover,

if \( N \) is large enough

∀ \( n \geq N \)

\[ |a_n - b_n| \leq \varepsilon \]

\[ |a_n - L| \leq \varepsilon \]

Convergence of Cauchy sequences follows from completeness.

More general def.
of completeness

is that Cauchy sequences converge.

Then In $\mathbb{R}^n$,
Cauchy sequences converge so $\mathbb{R}^n$
is complete.

If $(x^1, x^2, \ldots, x^n) = \overrightarrow{x}$

is Cauchy, then

each coord. is Cauchy

$x^i \rightarrow \overrightarrow{a}$ so conv.,

& each coord. converges.

$\Rightarrow$ to some $\overrightarrow{a}$

$x^i \rightarrow a^i$. 