Oct 14

HW
p. 84 # 4, 9

Higher order partials

Polynomial Approx.

\[ P(x, y) = \sqrt{2x^2 y + 3x y^2 + 4y} \]

If I didn't know coeff. I can find them by diff.

\[ \frac{\partial P}{\partial y} = 2x^2 + 6xy + \frac{1}{2} \]

\[ \frac{\partial P}{\partial y} (0, 0) = \]

\[ \frac{\partial P}{\partial y} (0, 0) = 0 + 0 + 0 = 0 \]
\[ \frac{\partial^2}{\partial x_1^2} P(x, y) = 6x \]

\[ \frac{\partial^3}{\partial y \partial x_1 \partial x_2} P(x, y) = 6 \]

\[ \frac{\partial^3}{\partial x_1 \partial y \partial x_2} P(y, 0) = 6 \]

\[ \frac{1}{2} \cdot 6 = 3 \text{ is the coefficient of } x^2 \]

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**Higher order partial derivatives:**

- \( f_{x^2} \)
- \( f_{x'y'} \)
- \( f_{xt} \)

\[ \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) \]

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\[ \frac{\partial}{\partial x} f_{x^2} = f_{x'y'} \]

\[ \frac{\partial}{\partial x} f_{xt} = \text{ pure partial} \]
\[ \text{Fact: } \text{The order of diff is immaterial for polynomials.} \]

If I want to approximate well by poly, it's mixed partials better not depend on order of diff.

Therefore suppose \( D \subseteq \mathbb{R}^2 \) open set \( \bar{a} = (a, b) \in D \)

\[ f : D \to \mathbb{R}, \quad \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x \partial y} \quad \text{all continuous at } \bar{a}. \]

Then

\[ \frac{\partial}{\partial x} \frac{\partial f}{\partial y} (\bar{a}) = \frac{\partial}{\partial y} \frac{\partial f}{\partial x} (\bar{a}). \]

Better than
Theorem Suppose \( D \subseteq \mathbb{R}^n \)
is open, \( f : D \to \mathbb{R} \)
for \( a \in D \). (\( f \) is a function)
and \( f \) is continuous at \( a \).
Then \( \partial f \) exists \( a \)
and \( \frac{\partial f}{\partial x} \)
is continuous at \( a \).

**Proof** \( \frac{\partial f}{\partial x} \) is continuous at \( a \).

\( a = (a_1, \ldots, a_n) \). For \( \mathbb{B}(a) \subseteq D \)

\[
\lim_{x \to a} \frac{f(x) - f(a)}{\mathbb{B}(a)} = \mathbb{E}(\mathbb{R})
\]

**Cont. of \( \frac{\partial f}{\partial x} \)** implies

\[
\lim_{R \to 0} \mathbb{E}(\mathbb{R}) = 0
\]
\[
\frac{f(a, b) - f(a+h, b) + f(a+h, b+h) - f(a, b+h)}{h \cdot k} \\
\]

\[
\frac{(f(a+h, b+h) - f(a+h, b)) - (f(a, b+h) - f(a, b))}{h \cdot k} \\
\]

\[
g'(x) = \frac{g'(a + h) - g'(a)}{h} \\
\]

\[
g' \text{ is differentiable on } [a, a + h] \\
\]

\[
\int_{a}^{b} f(c, b+h) \, dc \\
\]

\[
\frac{g'(c) \left( \frac{h}{h} \right)}{h} = c - b \forall h \in a \times a + h \\
\]

\[
- \int_{x}^{y} f(c, b+h) \, dc \\
\]
\[ h(x) = \frac{d}{dx} f(c, y) \]

\[ h = \frac{g(x+b) - g(x)}{b} \]

is defined on \([c, b+c]\]

and by MVT

\[ A = \rho(r) \]

\[ = 2 \int_c^d f(c, d) \]

since \((c, d) \text{ is in rect}\)

\[ \text{So} \]

\[ A = 2 \int_c^d f(c, d) \leq 3 \max_{x \in [0, 1]} |f(x)| \]
\[ f(x) = f(a, b) - f(a+h, b) + f(a+h, b+k) - f(\theta, b+k) \]

\[ \therefore \]

\[ f(a, b) - f(a+h, b+k) - f(a, b+k) \]

\[ \approx \]

\[ \frac{f(a+h, b) - f(a, b)}{h} \]

\[ \left| \frac{d}{dx} f(x) \right| \leq \varepsilon \left( \max(1, h) \right) \]

Take \( \lim_{h \to 0} \)

Set

\[ g(\gamma) = f(a+h, \gamma) - f(a, \gamma) \]

if \( \lim\) is diff. \( g \)

\[ \frac{1}{h} \]

\[ \lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h} \leq \varepsilon \left( \max(1, h) \right) \]
\[ a \xrightarrow{h \to 0} \mathcal{E}(h) \to 0 \]

So,
\[
\lim_{h \to 0} \frac{f(a + h, b) - b_y f(a, b)}{h} = \frac{\partial}{\partial x} f(a, b)
\]

\[
\frac{\partial}{\partial y} f(a, b) = \frac{\partial}{\partial x} f(a, b)
\]