Dec 2

Content $\alpha$ vs Measure $\Omega$

Note: If $C$ is rectifiable of $\alpha$-measure then it is of $\alpha$-content.

Int. in 2-dim.

Direct product of sets

$A \times B$ sets.

$A \times B = \sum (a, b) ; a \in A$.

$I, J$ — intervals in $\mathbb{R}$
\[ I \times J = \text{Rectangle} \]
\[ [a, b] \times [c, d] \]
\[ \text{Partitions} - \]
\[ P_i - \text{part. of } I \]
\[ P_j - \text{part. of } J \]
\[ P_i \times P_j \]
\[ P = \prod_{i} I_i \times J_j : \bigcup_{i,j} P_{i,j} \]
\[ = \sum_{i,j} R_{i,j} \]

Def. upper & lower sums

\[ \overline{S}(f, P) = \sum_{i,j} M_{i,j} \cdot \text{Area}(R_{i,j}) \]
\[ S(f, P) = \sum_{i,j} \min_{i,j} \text{Area}(B_{ij}) \]

\[ U(f) = \{ \text{all upper sums} \} \]
\[ L(f) = \{ \text{all lower sums} \} \]

\[ I(f) = \inf_{B \subseteq J} I_B(f) \]
\[ f \in \mathcal{F}(B) \]

\[ I(f) = \sup_{B \subseteq J} I_B(f) \]
\[ I_B(f) \geq I_B(f) \]

If \[ I_B(f) = \inf_{B \subseteq J} I_B(f) \]

Then we say \( f \) is \( R. \text{int} \) on \( B \).
\[ \int_{A} f \, dA = \left( \int_{a}^{b} \int_{c}^{d} f(x, y) \, dx \, dy \right) \]

\[ D \text{ Mesh}(P) = \max \{ \text{Mesh}P_1, \text{Mesh}P_2 \} \]

**Thm** If \( f \) is bounded, then 
\[ 0 < \varepsilon \rightarrow \exists \delta > 0 \]

\[ f \text{ Mesh}(P) < \delta \]

then 
\[ 0 \leq f(P) - \frac{1}{|P|} \leq \frac{f(P)}{\varepsilon} < \varepsilon \]

\[ 0 \leq \int_{B} f(x) - \int_{P} f(x) \leq \varepsilon, \]

**Thm** If \( f \) is bounded on the disc of \( f \) have content 0, then \( f \) is integrable.
Put $A$ in a box! = Characteristic or Indicator function

\[
X_A : I_A \quad I_A
\]

\[
X_A(x) = \begin{cases} 
1, & x \in A \\
0, & x \notin A 
\end{cases}
\]

\[
\int_B f dA = \int_B X_A f dA
\]
Lemma \( A \subseteq \mathbb{R}^2 \),
\[
\text{dist} \left( X_A \right) = \partial A.
\]

pf. \( \pi \in \partial A \), \exists \: \chi \in A^c, \chi \rightarrow \pi

\[\text{but} \quad \lim_{\chi \rightarrow \pi} X_A(\chi) = 1\]
\[\lim_{\chi \rightarrow \pi} X_A(\chi) = 0\]

\[
X \in A^{-1}, \quad J \geq 0
\]
\[B_r(x) \subseteq A, \quad \text{once } \frac{\|x - \delta\|}{r} < 1\]
\[
X_A(x) = X_A(\delta)
\]

\[X \in (A^c)^{-1}, \quad \forall \delta \]
\[
\Rightarrow \exists \: r > 0
\]
Then suppose $A \subseteq \mathbb{R}^2$ is bounded. Then $X_A$ is integrable iff $\int_A$ has content 0.

Use $\int_A = \text{cpt}$.

Area

We say $A$ is "measurable" if $X_A$ is integrable i.e. $\int_A$ has 0-content.

\[
B_r(x) \subseteq A^c.
\]
$$S(P, X_A) = \left( \text{sum of areas of all rect. that intersect } A \right)$$

$$\overline{S}(P, X_A) = \left( \text{sum of areas of all rect. cont. in } A \right)$$

$$I_B(X_A) - \text{outer area of } A,$$

$$\overline{I}_B(X_A) - \text{inner area of } A,$$